Calculating Stress Intensity Factor for Small Edge Radial Cracks on an Orthotropic Thick-walled Cylinder Subjected to Internal Pressure using the Average Stress

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In this paper, the problem of calculating the stress intensity factor (SIF) for an orthotropic thick-walled cylinder with a small radial crack subject to internal pressure is considered. The crack is assumed to be an edge crack on the external radius of the cylinder. The stress intensity factor is calculated by superposition of an uncracked cylinder with uniform stress distribution and a cylinder with a small radial crack. To calculate the stress distribution in the small radial crack, basic assumptions of the Linear Elastic Theory have been used. Due to the small length of the crack, an average stress method with a proper weight function is then used to evaluate the stress intensity factor of the assumed crack problem. Simplifying the proposed formula for orthotropic cylinder to the isotropic one, the results are validated against data given in the literature search.

Keywords: Small Radial Crack, Thick-Walled Cylinder, Stress Intensity Factor, Orthotropic Cylinder

1 Introduction

Brittle fracture is considered a serious threat to structures, particularly important structures such as aircrafts, bridges, pressure vessels, etc. Problems associated with cylindrical geometries having small cracks have wide applications in industry and many researches have been done on these problems. Shannon's work [1] is as an example of such works, where the stress intensity factor is calculated for a radial crack on a thick-walled cylinder with two cracks on the two ends of the diagonal of the cylinder, for various radial ratios using the finite element method.

Delale and Erdogan [2] presented the exact formulation of the plane elasticity problem for a hollow cylinder or a disk with a radial crack, and considered all three cases of the crack being

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on the external edge, internal edge, and embedded crack in their work. Sekine and Kuizumi [3] solved the problem of stress intensity factor for an embedded crack on a homogeneous orthotropic thick-walled cylinder subject to internal pressure, using the dislocation method. Ma et al. [4] applied the weight function method to obtain the stress intensity factor of a hollow cylinder. Pastrama and de Castro [5] exploited the fact that if a complete solution (the crack face displacement and the stress intensity factor) to a crack problem for one loading system is known, then the solution for the stress intensity factor for the same cracked configuration, but with any other loading, may be obtained directly from the known solution. Shlyannikov et al. [6] used the finite element method in two and three dimensions to study the fatigue crack growth and calculated the stress-strain relations and stress intensity factor of turbine disks. Wang [7] used the finite element method to calculate the dynamic stress intensity factor in a cracked thick-walled cylinder, presenting interesting computational patterns. Papadopoulos et al. [8] calculated the stress intensity factor for first crack mode on a reinforced crack, using the finite element method and the ANSYS FE program. Mahbadi [9] considered stress intensity factors of rotating hollow cylinders and disks made of functionally graded material (FGM) with small radial cracks, using the average stress method.

In this paper, the stress intensity factor for an orthotropic thick-walled cylinder with a small radial crack on its external edge, subject to uniform internal pressure is estimated, using the average stress of the cylinder along the crack length. The simple equations proposed here predict the stress intensity factor for the described geometrical configuration with engineering precision. These relations are obtained by combining the weight function and average stress methods. In the validation section, the relations from this paper are validated against relations obtained from the literature.

2 Mathematical Formulation

Consider an orthotropic hollow thick-walled cylinder with internal radius r_i and external radius r_o, having a radial crack of length a on the external edge, subject to internal and external pressures P_i and P_o, respectively. The principle axis of the fibers in this cylinder make angle θ with the axis of the cylinder. First, the radial and hoop stresses of the uncracked thick-walled cylinder is calculated. Then, using the superposition method, the stress intensity factor of the crack is determined. To this end, consider the uncracked cylinder as in Figure (1).

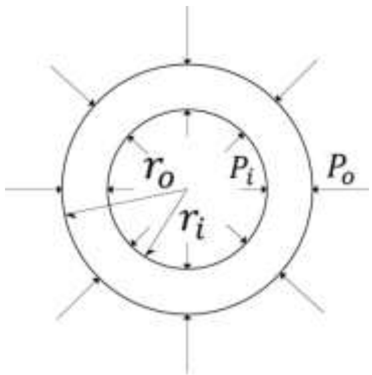

Figure 1 Isotropic thick-walled cylinder with internal and external radii r_i and r_o, and internal and external pressures P_i and P_o, respectively.

Assuming axial symmetry, the equilibrium equation in the r direction of polar coordinates becomes:

$$\frac{\partial\sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{1}$$

The stress-strain and strain-displacement relations lead to the equation below:

$$\begin{bmatrix}\sigma_r\\ \sigma_\theta\\ \tau_{r\theta}\end{bmatrix} = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26}\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix}\begin{bmatrix}\frac{du}{dr}\\ \frac{u}{r}\\ 2\varepsilon_{r\theta}\end{bmatrix} \tag{2}$$

Where $[\bar{Q}]$ is the stiffness matrix for orthotropic materials in plane stress conditions and in an arbitrary direction. Now, substituting the strain-displacement relations into the stress-strain equations and plugging them into the equilibrium equation, Euler's differential equation is obtained as follow:

$$\bar{Q}_{11}\frac{d^2u}{dr^2} + \frac{1}{r}(\bar{Q}_{12} + \bar{Q}_{11} - \bar{Q}_{12})\frac{du}{dr} + \frac{1}{r^2}(-\bar{Q}_{12} + \bar{Q}_{12} - \bar{Q}_{22})u = 0 \tag{3}$$

Solving the above equation, the displacement equation is obtained:

$$u = Ar^{-\sqrt{\mathrm{M}}} + Br^{\sqrt{\mathrm{M}}} \tag{4}$$

where

$$M = \frac{\bar{Q}_{22}}{\bar{Q}_{11}} \tag{5}$$

The coefficients A and B are determined using the boundary conditions described at the beginning of this section:

$$\mathrm{A} = \frac{-\mathrm{P}_o \mathrm{r}_o{}^{\sqrt{\mathrm{M}}+1} - B(\bar{Q}_{12} + \bar{\mathrm{Q}}_{11}\sqrt{\mathrm{M}})\mathrm{r}_o{}^{2\sqrt{\mathrm{M}}}}{(\bar{Q}_{12} - \bar{\mathrm{Q}}_{11}\sqrt{\mathrm{M}})} \tag{6}$$

$$\mathrm{B} = \frac{\mathrm{P}_o \mathrm{r}_o{}^{\sqrt{\mathrm{M}}+1}\mathrm{r}_i{}^{-\sqrt{\mathrm{M}}-1} - P_i}{(\bar{Q}_{12} + \bar{\mathrm{Q}}_{11}\sqrt{\mathrm{M}})(\mathrm{r}_i{}^{\sqrt{\mathrm{M}}-1} - \mathrm{r}_o{}^{2\sqrt{\mathrm{M}}}\mathrm{r}_i{}^{-\sqrt{\mathrm{M}}-1})} \tag{7}$$

Finally, for the radial and hoop stresses we get:

$$\sigma_\theta = \mathrm{A}\left(\bar{Q}_{22} - \bar{Q}_{12}\sqrt{\mathrm{M}}\right)\mathrm{r}^{-\sqrt{\mathrm{M}}-1} + B(\bar{Q}_{22} + \bar{Q}_{12}\sqrt{\mathrm{M}})\mathrm{r}^{\sqrt{\mathrm{M}}-1} \tag{8}$$

$$\sigma_r = \mathrm{A}\left(\bar{Q}_{12} - \bar{\mathrm{Q}}_{11}\sqrt{\mathrm{M}}\right)\mathrm{r}^{-\sqrt{\mathrm{M}}-1} + B(\bar{Q}_{12} + \bar{\mathrm{Q}}_{11}\sqrt{\mathrm{M}})\mathrm{r}^{\sqrt{\mathrm{M}}-1} \tag{9}$$

Now according to Figure (2), the stress intensity factor for a cracked cylinder with internal pressure would be equivalent to the stress intensity factor for a cracked cylinder with the stress distribution σ_θ on the crack surface.

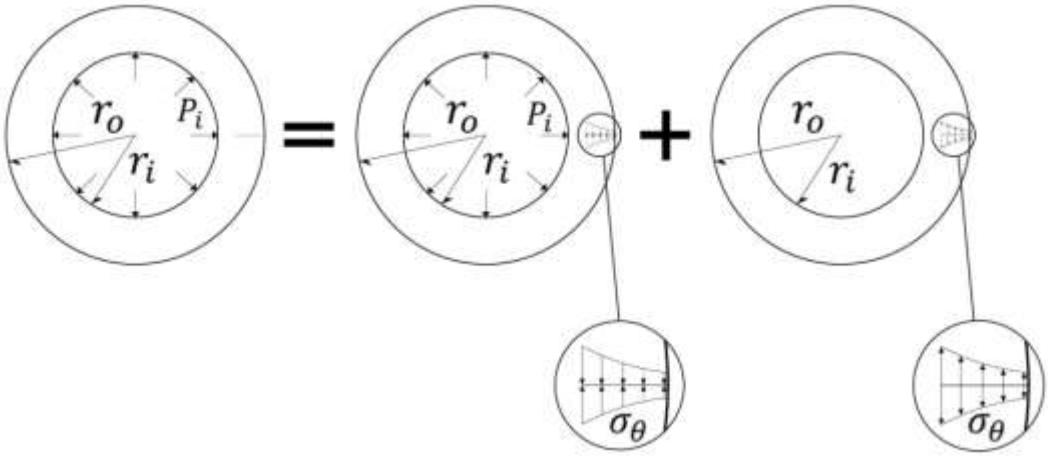

Figure 2 The superposition method used in solving the crack problem

On the other hand, if we take the crack length to be small, we can consider the stress intensity factor to be approximately equal to the stress intensity factor of a cracked cylinder whose crack edge is subject to the uniform stress σ_{ave}, the average of the circumferential stress along the edge of the crack (see Figure 3).

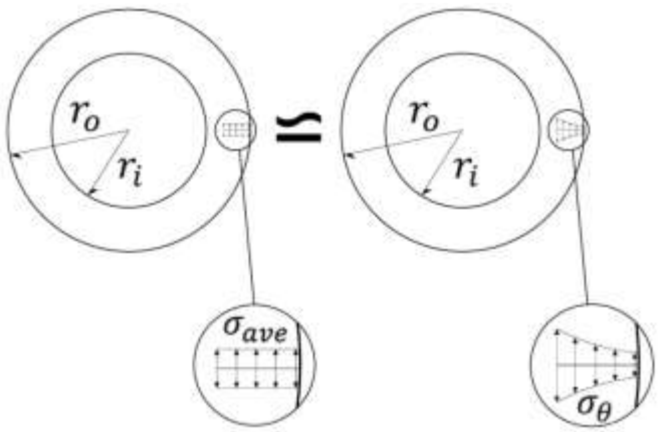

Figure 3 Equalizing the crack problem with non-uniform stress distribution and the crack problem with the average stress distribution

Therefore, taking the average of σ_θ along the crack length, the average stress on the edge of the crack is calculated thus:

$$\sigma_{ave} = \frac{1}{a}\int_{r_o-a}^{r_o} \sigma_\theta dr \tag{10}$$

Hence, the average stress would be:

$$\sigma_{ave} = \frac{1}{a}\left[\frac{A\left(\bar{Q}_{22}-\bar{Q}_{12}\sqrt{M}\right)\left[r_o^{-\sqrt{M}}-(r_o-a)^{-\sqrt{M}}\right]}{-\sqrt{M}} + \frac{B(\bar{Q}_{22}+\bar{Q}_{12}\sqrt{M})[r_o^{\sqrt{M}}-(r_o-a)^{\sqrt{M}}]}{\sqrt{M}}\right] \tag{11}$$

Since the stress intensity factor of the first mode is given by:

$$K_I = F_c\sigma_0\sqrt{\pi a} \tag{12}$$

Where σ_0 is the stress distribution at the crack tip and F_c is a function that determines the geometrical coefficient of the crack, in the given problem, stress intensity factor can therefore be written as:

$$K_I = F_c \sigma_{ave} \sqrt{\pi a} \tag{13}$$

Having the appropriate geometrical coefficient for the problem at hand, one can be able to approximately calculate the stress intensity factor. Ma et al. [4], Pastrama and de Castro [5], and Andrasic and Parker [10] have calculated stress intensity factors on the edges of the crack for edge cracks on a cylinder subject to uniform traction stress. Therefore, having the stress intensity factor, the geometrical coefficient of the crack is calculated using Eq. (12) as follows:

$$F_c = \frac{K_I}{\sigma_{ave}\sqrt{\pi a}} \tag{14}$$

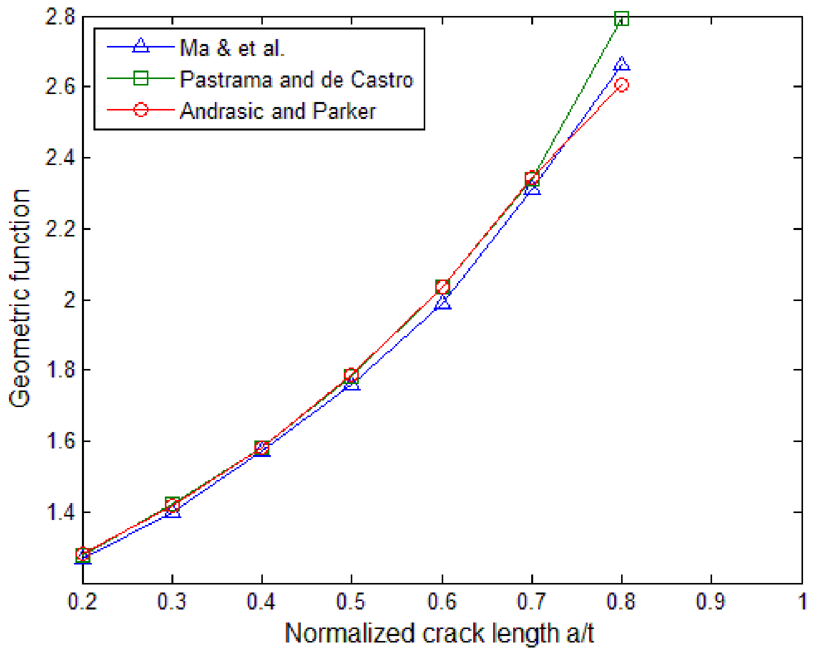

Figure 4 Comparison of various geometrical coefficients

Geometrical coefficients based on the results by Ma et al., De Castro and Pastrama, and Andrasic and Parker are given in Fig. (4), with $\nu = 0.3$, $a/t = 0.5$, and $r_i/r_o = 0.5$.

Now, the stress intensity factor for the problem mentioned in this paper is calculated, choosing the weight function proposed by Ma et al. which demonstrates good agreement with the other weight functions, and has been written as:

$$F_c = \frac{2\sqrt{2}}{\pi} + \frac{2}{3}A^1(a_0, W) + \frac{2}{5}A^2(a_0, W) \tag{15}$$

In the equation above, the coefficients A^1 and A^2, which are functions of dimensionless crack length and the ratio of internal and external coefficients $W = \frac{r_i}{r_o}$, are calculated.

$$A^i(a_0, W) = \sum_{m=1}^{5}\sum_{n=1}^{6} C_{mn}^i \left(\frac{a_0}{1-W}\right)^{n-1} W^{m-1}, \quad i=1,2 \tag{16}$$

Numerical values for the coefficients C_{mn}^1 and C_{mn}^2 for a hollow cylinder with a radial crack on the external surface are given in Tables (1) and (2), respectively. The first mode of stress intensity factor for the given problem will eventually be calculated from Eq. (17).

$$K_I = \left(\frac{2\sqrt{2}}{\pi} + \frac{2}{3}A^1(a_0, W) + \frac{2}{5}A^2(a_0, W)\right)\sigma_{ave}\sqrt{\pi a} \tag{17}$$

Table 1 Coefficients C_{mn}^1 for a radial crack on the external edge of the thick-walled cylinder

	N					
M	1	2	3	4	5	6
1	2.5501031E-1	0	0	0	0	0
2	5.4080143E+1	-5.6152875E+2	2.2443203E+3	-4.3442256E+3	4.0749150E+3	-1.4852133E+3
3	-5.0961920E+2	5.3548418E+3	-2.1458539E+4	4.1624289E+4	-3.9122293E+4	1.4290093E+4
4	1.2670717E+3	-1.3267471E+4	5.3177668E+4	-1.0316833E+5	9.7056742E+4	-3.5495023E+4
5	-9.2447144E+2	9.6765908E+3	-3.8768777E+4	7.5214820E+4	-7.0813688E+4	2.5934543E+4

Table 2 Coefficients C_{mn}^2 for a radial crack on the external edge of the thick-walled cylinder

	N					
M	1	2	3	4	5	6
1	1.3602442E-1	0	0	0	0	0
2	-6.2564392E+1	6.5881152E+2	-2.6154922E+3	5.0134692E+3	-4.6529854E+3	1.6780341E+3
3	6.0184979E+2	-6.2566255E+3	2.4853787E+4	-4.7702477E+4	4.4335598E+4	-1.6012451E+4
4	-1.4538805E+3	1.5061571E+4	-5.9776449E+4	1.1467289E+5	-1.0651307E+5	3.8437199E+4
5	1.0249813E+3	-1.0583187E+4	4.1929215E+4	-8.0310359E+4	7.4472859E+4	-2.6815914E+4

3 Results and discussion

In this section, first, applying isotropic conditions to the derived equations, the stress intensity factor for a radial crack on the external edge of an isotropic cylinder is calculated and the validity of the results is checked. Then, stress intensity factors in various cases for the orthotropic material Graphite Epoxy with similar geometries are compared.

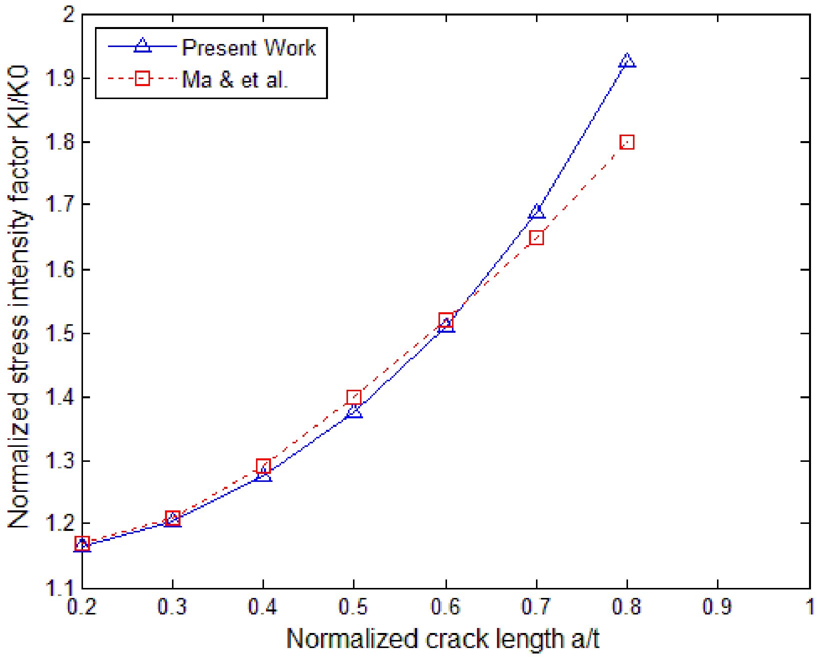

Figure 5 Dimensionless stress intensity factor with respect to K_0, for an edge crack of length a, on the external surface of a thick-walled cylinder subject to uniform internal pressure P_0, as a function of the dimensionless crack length with respect to $t = r_o - r_i$

In Figure (5), the stress intensity factors are normalized with respect to $K_0 = P_0\sqrt{\pi a}$, where $\nu = 0.3$, $a/t = 0.5$, and $r_i/r_o = 0.5$. The result show that the formula proposed in this paper is close to the result by Ma et al., even for large cracks. The maximum difference is less than 6.5 %.

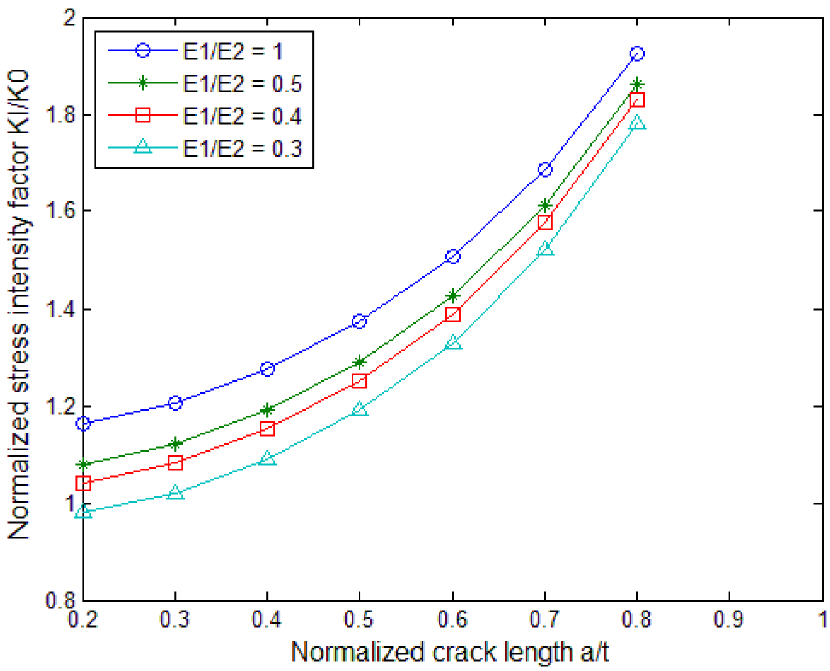

Figure 6 Dimensionless stress intensity factor with respect to K_0 for an edge crack of length a, on the external surface of a thick-walled cylinder subject to uniform internal pressure P_0, as a function of the dimensionless crack length with respect to $t = r_o - r_i$, at various values for $E_1/E_2 \leq 1$

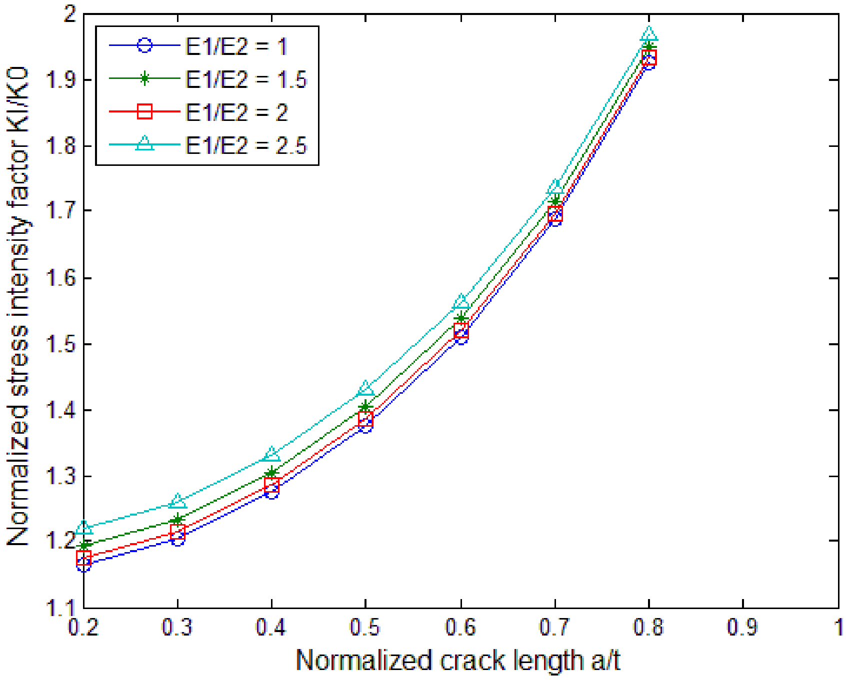

Figure 7 Dimensionless stress intensity factor with respect to K_0 for an edge crack of length a, on the external surface of a thick-walled cylinder subject to uniform internal pressure P_0, as a function of the dimensionless crack length with respect to $t = r_o - r_i$, at various values for $E_1/E_2 \geq 1$

The plots in Figures (6) and (7) are obtained from the orthotropic relations derived in this paper, evaluated at a 0 degree angle, with the parameters $a/t = 0.5$, and $r_i/r_o = 0.5$. Glancing at these plots it is evident that as the ratio of the elasticity module approaches unity, the material behavior approaches the isotropic case. Also, it may be seen that by increase in ratio of E_1/E_2 for values of $E_1/E_2 > 1$, the stress intensity factor will increase, while decreasing the ratio of E_1/E_2, for the values of $E_1/E_2 < 1$, the stress intensity factor will decrease.

Now using the derived relations for orthotropic material, the stress intensity factor for an edge crack on an orthotropic thick-walled cylinder made of Graphite Epoxy is evaluated. In computing the numerical value of the stress intensity factor, the following parameters have been used: $E_1 = 181\ GPa$, $E_2 = 10.3\ GPa$, $\nu_{12} = 0.28$, and $G_{12} = 7.17\ GPa$.

Figure (8) shows that: as the angle of fibers in the orthotropic material increases, the stress intensity factor decreases, and the rate of this decrease is lower at degrees closer to zero. The parameters in this case are $a/t = 0.5$, and $r_i/r_o = 0.5$. In Figure (9), the decrease of $W = r_i/r_o$ in the orthotropic thick-walled cylinder made of Graphite Epoxy leads to a decrease in the stress intensity factor. Here, the formula is evaluated at a 0 degree angle of fiblers.

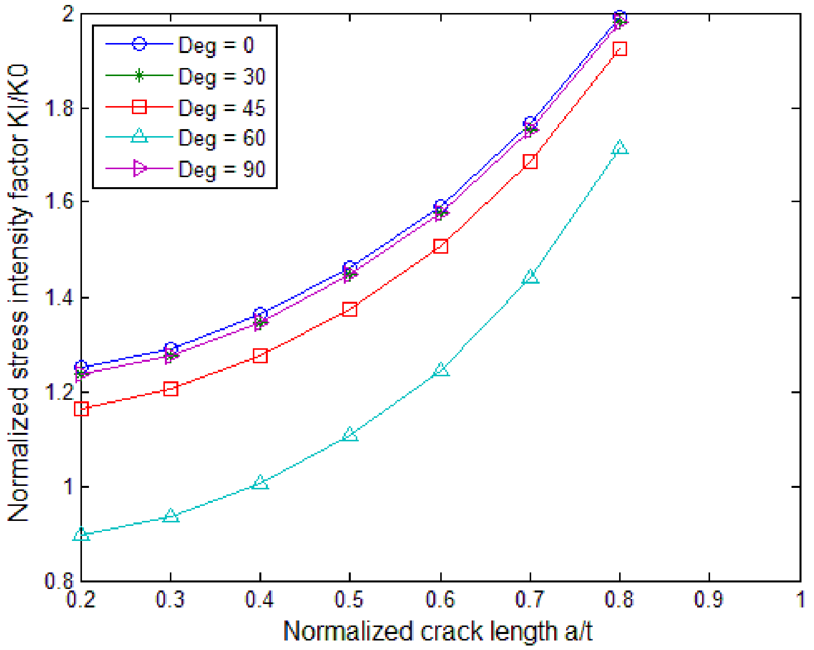

Figure 8 Dimensionless stress intensity factor with respect to K_0 for an edge crack of length a, on the external surface of a thick-walled cylinder subject to uniform internal pressure P_0, as a function of the dimensionless crack length with respect to $t = r_o - r_i$, at various fiber angles

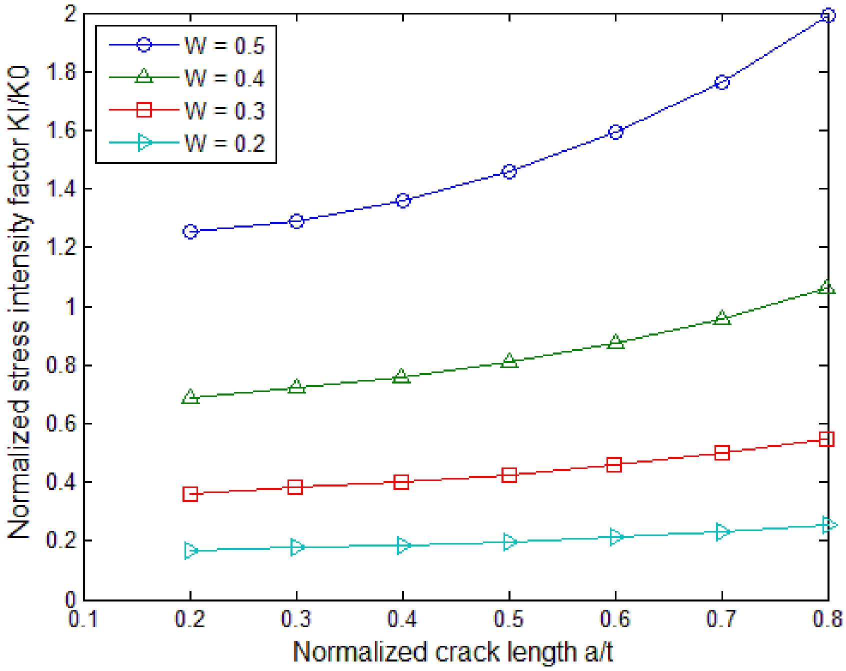

Figure 9 Dimensionless stress intensity factor with respect to K_0 for an edge crack on the external surface of a thick-walled cylinder subject to uniform internal pressure P_0, as a function of the dimensionless crack length with respect to $t = r_o - r_i$, at various radial ratios $W = r_i/r_o$

4 Conclusion

In this research, an approximate formula is proposed for computing the stress intensity factor of radial cracks on an orthotropic cylinder. A comparison between results based upon this formula and results from numerical methods in other sources shows that the relations derived for small cracks can satisfactorily predict the stress intensity factor for small cracks. Also, the results obtained for cracks with medium length show good agreement with results from numerical methods. But since this is an approximate method, the error increases for medium and large cracks. The plotted curves demonstrate that the stress intensity factor decreases as the fiber angle increases. Also, a decrease in the ratio of internal to external radii, or more generally an increase in the thickness of the cylinder would lead to a decrease in the stress intensity factor. Another conclusion is that, increase in difference of axial and transverse modulus of elasticity will affect to stress intensity factor of the mentioned problem, in such a way that increase in ratio of axial modulus of elasticity to transverse modulus of elasticity of the orthotropic cylinder with zero fibers angle, will increase the stress intensity factor of the mentioned problem.

Nomenclatures

a: Crack Length
C^1_{mn}: m, n-th Coefficient in Table 1
C^2_{mn}: m, n-th Coefficient in Table 2
h_I: Weight Function for Mode I
K: Stress Intensity Factor Relative to Average Stress
K_0: Stress Intensity Factor
P_i: Internal Pressure
P_o: External Pressure
$\bar{Q}_{mn}$: m, n-th Component of Stiffness Matrix
r_i: Internal Radius
r_o: External Radius
t: Cylinder Thickness
u: Displacement Function
W: Internal to External Radius Ratio

Greek Symbols

$\varepsilon_{r\theta}$: Strain in Polar Coordinates
ν: Poisson Ratio
σ: Stress
σ_{ave}: Average Stress
σ_θ: Hoop Stress in Polar Coordinates
$\tau_{r\theta}$: Shear Tension in Polar Coordinates

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چکیده

در این مقاله، ضریب شدت تنش، برای یک استوانه‌ی جدار ضخیم ارتوتروپ، با یک ترک کوچک شعاعی محاسبه شده است. فرض بر این است که ترک لبه‌ای و روی شعاع خارجی استوانه قرار دارد و استوانه تحت فشار داخلی یکنواخت است. در این مقاله ضریب شدت تنش با استفاده از برهمنهی یک استوانه‌ی بدون ترک تحت فشار داخلی یکنواخت و یک استوانه با ترک کوچک شعاعی تحت تنش یکنواخت عمود بر سطح ترک محاسبه شده است. به منظور محاسبه‌ی توزیع تنش در ترک کوچک شعاعی، معادلات حاکم بر تئوری الاستیک خطی برای تعیین تنش‌ها حل شده‌اند. آنگاه با توجه به کوچک بودن طول ترک از روش تنش متوسط به منظور محاسبه‌ی توزیع تنش یکنواخت بر روی لبه‌ی ترک استفاده شده است. نهایتا با انتخاب تابع وزنی مناسب، ضریب شدت تنش برای ماده‌ی ارتوتروپ گرافیت اپوکسی تعیین شده است. با ساده کردن روابط ارتوتروپ به حالت ایزوتروپ، نتایج اعتبارسنجی شده‌اند.