ISME

Elastoplastic Analysis of Functionally Graded Beams under Mechanical Loads

H. Mahbadi *
Assistant Professor

Elastic-plastic behavior of a beam made of functionally graded material is investigated in this work. The beam is subjected to the constant axial and bending loads and the critical values of these loads for yield, collapse and elastic-plastic conditions are obtained. The variation of elastic modulus and yield strength through the height of the beam is determined with an exponential rule. The perfect plastic curve is used to model the plastic behavior of the beam. The interaction diagrams between the bending moment and axial load are obtained for both of the yield and collapse conditions. The effect of power law function on yield and collapse loads is estimated. The results are reduced to the homogeneous beam and validated with data given in the literature search.

Keywords: FGM, Collapse load, Yield load, Mechanical load, Plastic analysis, Beam

1 Introduction

Functionally graded materials (FGMs) are a class of materials which are characterized by gradual variation in composition, micro structure and material properties in pre-determined profile. Accordingly, while FGM benefits the advantage of composite materials in efficient designing of structures, they eliminate various shortcomings resulting from stepwise property mismatch inherent in piecewise homogeneous composite medias [1, 2]. The FGM initially was designed as thermal barrier in spacecrafts, but due to advantage and technical potential of these materials, they have been widely used in transportation, energy, electronics and biomedical engineering [3, 4].

The mathematical modeling of these materials in order to study the behavior of the structures under thermo mechanical loads in elastic region is developed by researchers. The general solution presented for the one-dimensional steady-state thermal and mechanical stresses in a hollow thick sphere made of functionally graded material by Eslami and et al., [5] and study of the coupled thermoelasticity of an FG cylinder subjected to the thermal shock load by Bahtui and Eslami [6] are examples of such works. On the other hand, the mathematical modeling of structures made of functionally graded materials in plastic region is less developed due to the literature search performed by the author. Shakedown analysis of FG Bree plates subjected to the thermomechanical loading performed by Peng and et al. [7, 8] and elastic-plastic analysis of two-dimensional functionally graded materials under thermal loading given in [9] are among such works.

* Assistant Professor, Department of Mechanical Engineering, Islamic Azad University, Central Tehran Branch, Tehran, Iran, h_mahbadi@iauctb.ac.ir

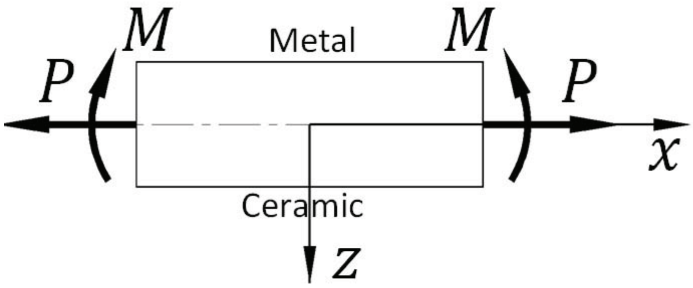

Figure 1 Assumed geometry and coordinate system of the FGM beam

The plastic collapse load of a beam made of functionally graded material is analyzed in this work. The beam is subjected to the constant axial and bending loads. The variation of elastic modulus and yield strength through the height of the beam is determined with an exponential rule. The perfect plastic stress-strain curve is used to evaluate the plastic strains. The elastic boundary and collapse load boundaries are identified by solution of the corresponding equations. The effect of power law function on the yield, collapse and elastic-plastic behavior of the beam is investigated. The results are reduced to the homogeneous beam and validated with data given in the literature search.

2 Mathematical Concepts

A beam of rectangular section with the height h and thickness of b is considered under axial load P on its neutral axis and a bending moment M in the xz plane. The length of the beam is laid through the $x-$ axis and its height is laid through the $z-$ axis with downward positive direction, as shown in Fig. (1). The directions shown in this figure for bending moment M and axial load P are positive. The opposite directions of axial and bending loads are represented by $-M$ and $-P$, respectively. The beam material at $z = -h/2$ and $z = h/2$ is fully metal and fully ceramic, respectively, while the material between these two surfaces is graded and so the material parameters are spatial function of z. Assuming the power law function [10] for the material parameters, the elastic modulus $E(z)$ and yield strength $S_y(z)$ of the beam obey the following equations:

$$E(z) = E_m + (E_c - E_m)\left(\frac{2z+h}{2h}\right)^n \tag{1}$$

$$S_y(z) = S_{ym} + (S_{yc} - S_{ym})\left(\frac{2z+h}{2h}\right)^n \tag{2}$$

where E_m and E_c are elastic modulus and S_{ym} and S_{yc} are the yield strength of the metal and ceramic, respectively. The total mechanical strain $\epsilon_x(z)$ at the beam cross section considering the yielding of the beam is:

$$\epsilon_x(z) = \epsilon_x^e(z) + \epsilon_x^p(z) \tag{3}$$

where $\epsilon_x^e(z)$ and $\epsilon_x^p(z)$ are elastic and plastic strain distribution in cross section of the beam, respectively. Assuming the perfect plastic behavior of the beam material in the plastic region, the stress distribution in the elastic and plastic regions are:

$$\sigma_x(z) = E(z) * \epsilon_x^e(z) \qquad in\ elastic\ region \tag{4}$$

$$\sigma_x(z) = S_y(z) \qquad in\ plastic\ region \tag{5}$$

The compatibility differential equation is:

$$\frac{d^2\epsilon_x(z)}{dz^2} = 0 \tag{6}$$

Equilibrium of the axial load P and bending moment M at the ends of the strip, results:

$$b\int_{-h/2}^{h/2} \sigma_x(z)dz = P \tag{7}$$

$$b\int_{-h/2}^{h/2} \sigma_x(z)zdz = M \tag{8}$$

In next sections, the beam is analyzed for three different conditions. In the first condition it is assumed that the beam begins to yield, wherein, the cross section of the beam is fully elastic. In the second condition, it is assumed that the cross section of the beam has been yielded but it is not fully plastic yet, wherein, the cross section of the beam is partly in elastic and partly in plastic regions. Finally, the third condition corresponds to collapse of the beam, wherein, the cross section of the beam is fully plastic. To this aim the compatibility equation (6) is solved and the constants of integrations are found for each case, separately. The solution of Eq. (6) is:

$$\epsilon_x(z) = c_0 + c_1 z \tag{9}$$

The constants of integrations are obtained using the equilibrium of axial and bending loads at the ends of the strip given by Eqs. (7) and (8). Also, the place of neutral axis z_n is estimated by setting the axial total strain to zero:

$$z_n = -\frac{c_0}{c_1} \tag{10}$$

3 Beam at Beginning of the Yield

In this condition the cross section of the beam is fully elastic. Imposing the conditions (7) and (8), the constant of integrations are:

$$c_0 = \frac{B_2^e}{B_0^e B_2^e - B_1^{e2}}(\frac{P}{b}) - \frac{B_1^e}{B_0^e B_2^e - B_1^{e2}}(\frac{M}{b}) \tag{11}$$

$$c_1 = -\frac{B_1^e}{B_0^e B_2^e - B_1^{e2}}(\frac{P}{b}) + \frac{B_0^e}{B_0^e B_2^e - B_1^{e2}}(\frac{M}{b}) \tag{12}$$

Wherein, definition of B_i^e's are:

$$B_0^e = \int_{-h/2}^{h/2} E(z)dz = E_m h + (E_c - E_m)\frac{h}{n+1} \tag{13}$$

$$B_1^e = \int_{-h/2}^{h/2} E(z)zdz = (E_c - E_m)\frac{nh^2}{2(n+1)(n+2)} \tag{14}$$

$$B_2^e = \int_{-h/2}^{h/2} E(z)z^2dz = E_m\frac{h^3}{12} + (E_c - E_m)\frac{(n^2+n+2)h^3}{4(n+3)(n+2)(n+1)} \tag{15}$$

Substituting the constants of integrations into Eq. (10), the height of the neutral axis z_n^e at beginning of the yield is:

$$z_n^e = -\frac{B_2^e P - B_1^e M}{B_1^e P - B_0^e M} \tag{16}$$

Depending to the values of yield stress and elastic modulus at top and bottom of the beam, the yield begins at either $z_y = -h/2$ or $z_y = h/2$. Assuming equal tensile and compressive yield stress for both the metal and ceramic phase of the beam, the yield condition is:

$$\begin{aligned} E(z_y)(c_0 + c_1 z_y) &= S_y(z_y) \quad && when\ \epsilon_x > 0 \\ E(z_y)(c_0 + c_1 z_y) &= -S_y(z_y) \quad && when\ \epsilon_x < 0 \end{aligned} \tag{17}$$

Now, consider three different load cases corresponding to pure axial loading, pure bending loading and combination of axial and bending loads.

3.1 Yield Axial Load

Assuming the FG beam is subjected to pure tensile axial load, then strains are also tensile and the first condition of (17) results to yield axial load P_y which brings the beam up to the yield. That is:

$$P_y = b\frac{(B_0^e B_2^e - B_1^{e2})[S_{ym} + (S_{yc} - S_{ym})(z_y/h + 1/2)^n]}{(B_2^e - B_1^e z_y)[E_m + (E_c - E_m)(z_y/h + 1/2)^n]} \tag{18}$$

where due to the variation of the material property, yield point is at either $z_y = -h/2$ or $z_y = h/2$. To check this, one can obtain P_y for both $z_y = -h/2$ and $z_y = h/2$. The critical axial load which brings the beam up to the yield is minimum of these two values. For compressive axial loading, it is possible to repeat the same procedure with the second yield condition of Eq. (17).

3.2 Yield Bending Load

Now, assume the FG beam subjected to pure bending moment in direction as shown in Fig. (1). Due to the variation of material property, the yield begins from $z = -h/2$ wherein, the axial strain is compressive or $z = h/2$, wherein the axial strain is tensile. If yield begins from the compressive surface of the beam, then $z_y = -h/2$ and according to the second of Eq. (17), the yield bending moment M_y is:

$$M_y = b\frac{(B_0^e B_2^e - B_1^{e2})[-S_{ym} - (S_{yc} - S_{ym})(z_y/h + 1/2)^n]}{(-B_1^e + B_0^e z_y)[E_m + (E_c - E_m)(z_y/h + 1/2)^n]} \tag{19}$$

On the other hand, when the yield begins from the regions which the stress distribution is tensile, then $z_y = h/2$ and according to the second of Eq. (17), the yield bending moment is:

$$M_y = b\frac{(B_0^e B_2^e - B_1^{e2})[S_{ym} + (S_{yc} - S_{ym})(z_y/h + 1/2)^n]}{(-B_1^e + B_0^e z_y)[E_m + (E_c - E_m)(z_y/h + 1/2)^n]} \tag{20}$$

As described for the case of pure axial loading, in order to check that the yield begins from $z_y = -h/2$ or $z_y = h/2$, the yield bending moment is calculated from both of (19) and (20). Then the minimum of these values is considered as the yield bending moment and the yield begins from the surface where M_y is minimum.

3.3 Yield Axial and Bending Load

In this case the yield condition for combination of axial and bending load is obtained (i.e. $P \neq 0$ and $M \neq 0$). To this aim it is assumed that an axial load P less than P_y is applied to the structure, then the critical bending moment which brings the beam up to the yield point is evaluated. Depending on that, the yield begins at $z = -h/2$ or $z = h/2$ and at that point the strains are compressive or tensile. Four different cases are considered. When the yield begins from the point where the axial strain is compressive, the yield bending moment according to the second of Eq. (17) is:

$$M_y = \frac{b[B_0^e B_2^e - (B_1^e)^2][-S_{ym} - (S_{yc} - S_{ym})(Z_y/h + 1/2)^n]}{(-B_1^e + B_0^e Z_y)[E_m + (E_c - E_m)(z_y/h + 1/2)^n]} - (\frac{B_2^e - B_1^e z_y}{-B_1^e + B_0^e z_y})P \tag{21}$$

When the yield begins from the point where the axial strain is tensile, the yield bending moment according to the first of Eq. (17) is:

$$M_y = \frac{b[B_0^e B_2^e - (B_1^e)^2][S_{ym} + (S_{yc} - S_{ym})(Z_y/h + 1/2)^n]}{(-B_1^e + B_0^e Z_y)[E_m + (E_c - E_m)(z_y/h + 1/2)^n]} - (\frac{B_2^e - B_1^e z_y}{-B_1^e + B_0^e z_y})P \tag{22}$$

In each of Eqs. (21) and (22) the yield bending moment M_y should be calculated for two different values of $z_y = -h/2$ and $z_y = h/2$. The yield bending moment is minimum of these four calculated values and the yield begins from the point which the minimum value corresponds. In order to obtain the yield axial load P_y for a bending load M, less than M_y, one may solve Eqs. (21) and (22) to obtain P versus the bending moment.

4 Elastic-Plastic Analysis

In this section, the load values which are responsible to flow of the beam into the plastic region are obtained. Assuming the regions with height $z \leq z_{y1}$ and $z \geq z_{y2}$ are plastic, the integrals in conditions (7) and (8) are divided into three parts as follow:

$$\int_{-h/2}^{z_{y1}} \sigma_x(z)dz + \int_{z_{y1}}^{z_{y2}} \sigma_x(z)dz + \int_{z_{y2}}^{h/2} \sigma_x(z)dz = P/b \tag{23}$$

$$\int_{-h/2}^{z_{y1}} \sigma_x(z)zdz + \int_{z_{y1}}^{z_{y2}} \sigma_x(z)zdz + \int_{z_{y2}}^{h/2} \sigma_x(z)zdz = M/b \tag{24}$$

Substituting the stress distribution in elastic and plastic regions form (4) and (5) into Eqs. (23) and (24), the constants of integration are found. If the regions with $z \leq z_{y1}$ be compressive and the regions with $z \geq z_{y2}$ be tensile, then the solution for constants of integration is:

$$c_0 = \frac{(P/b - A_0' + A_0)B_2^{ep} - (M/b - A_1' + A_1)B_1^{ep}}{B_0^{ep} B_2^{ep} - (B_1^{ep})^2} \tag{25}$$

$$c_1 = \frac{-(P/b - A_0' + A_0)B_1^{ep} + (M/b - A_1' + A_1)B_0^{ep}}{B_0^{ep} B_2^{ep} - (B_1^{ep})^2} \tag{26}$$

The place of neutral axis z_n^{ep} for the case where the beam is partly in plastic region is:

$$z_n^{ep} = -\frac{(P/b - A_0' + A_0)B_2^{ep} - (M/b - A_1' + A_1)B_1^{ep}}{-(P/b - A_0' + A_0)B_1^{ep} + (M/b - A_1' + A_1)B_0^{ep}} \tag{27}$$

On the other hand, if the regions with $z \leq z_{y1}$ be tensile and the regions with $z \geq z_{y2}$ be compressive, then the constants of integration are:

$$c_0 = \frac{(P/b + A_0' - A_0)B_2^{ep} - (M/b + A_1' - A_1)B_1^{ep}}{B_0^{ep} B_2^{ep} - (B_1^{ep})^2} \tag{28}$$

$$c_1 = \frac{-(P/b + A_0' - A_0)B_1^{ep} + (M/b + A_1' - A_1)B_0^{ep}}{B_0^{ep} B_2^{ep} - (B_1^{ep})^2} \tag{29}$$

and the place of neutral axis is:

$$z_n^{ep} = -\frac{(P/b + A'_0 - A_0)B_2^{ep} - (M/b + A'_1 - A_1)B_1^{ep}}{-(P/b + A'_0 - A_0)B_1^{ep} + (M/b + A'_1 - A_1)B_0^{ep}} \tag{30}$$

In Eqs. (25) through (30), the values of A_i, A'_i and B_i^{ep} are defined using the following integrals:

$$A_0 = \int_{-h/2}^{z_{y1}} S_y(z)dz \quad A_1 = \int_{-h/2}^{z_{y1}} S_y(z)zdz \tag{31}$$

$$A'_0 = \int_{z_{y2}}^{h/2} S_y(z)dz \quad A'_1 = \int_{z_{y2}}^{h/2} S_y(z)zdz \tag{32}$$

$$B_0^{ep} = \int_{z_{y1}}^{z_{y2}} E(z)dz \quad B_1^{ep} = \int_{z_{y1}}^{z_{y2}} E(z)zdz \quad B_2^{ep} = \int_{z_{y1}}^{z_{y2}} E(z)z^2dz \tag{33}$$

Now, the values of axial and bending loads which results the regions bellow the z_{y1} and above the z_{y2} flow to the plastic region, is calculated for three different cases of pure axial load, pure bending load and their combination.

4.1 Elastic-Plastic Axial Load

Depending on the material properties, the imposed pure axial load may results to beginning of the yield from $z = -h/2$ or $z = h/2$. If sufficient axial load applied to the FG beam and yield begins from $z = -h/2$, then in regions with $z \leq z_{y1}$ the strain is tensile and stress is equal to yield stress of the beam. For this case, substituting $z_y = z_{y1}$ in first of the Eq. (17) the critical axial load P_{cr} which brings the regions with $z \leq z_{y1}$ up to the yield is:

$$P_{cr} = b\frac{S_y(z_1)}{E(z_1)}\frac{B_0^{ep}B_2^{ep} - (B_1^{ep})^2}{B_2^{ep} - B_1^{ep}z_1} + b\frac{(-A'_0 + A_0)(B_2^{ep} - B_1^{ep}z_1) + (-A'_1 + A_1)(-B_1^{ep} + B_0z_1)}{B_2^{ep} - B_1^{ep}z_1} \tag{34}$$

It should be mentioned that in Eq. (34), if the edge of the beam with $z_2 = h/2$ does not begin to yield, then: $z_{y2} = h/2$ and so, Eq. (32) results that coefficients A'_0 and A'_1 are zero. If the axial load increased up to the value that also brings the regions with $z \geq z_{2y}$ to the yield, then one may use the second of Eq. (17) to calculate z_{y1} corresponding to z_{y2}. That is:

$$z_{y1} = \frac{B_2^{ep}}{B_1^{ep}} - (\frac{B_2^{ep}}{B_1^{ep}} - z_{y2})\frac{N_1}{N_2} \tag{35}$$

Wherein, N_1 and N_2 are:

$$\begin{aligned} N_1 = &\frac{S_y(z_{y1})}{E(z_{y1})}(B_0^{ep}B_2^{ep} - (B_1^{ep})^2) + (-A'_0 + A_0)(B_2^{ep} - B_1^{ep}z_{y1}) \\ &+ (-A'_1 + A_1)(-B_1^{ep} + B_0^{ep}z_{y1}) \end{aligned} \tag{36}$$

$$\begin{aligned} N_2 = -&\frac{S_y(z_{y2})}{E(z_{y2})}(B_0^{ep}B_2^{ep} - (B_1^{ep})^2) + (-A'_0 + A_0)(B_2^{ep} - B_1^{ep}z_{y2}) \\ &+ (-A'_1 + A_1)(-B_1^{ep} + B_0^{ep}z_{y2}) \end{aligned} \tag{37}$$

On the other hand, if the beam begins to yield from $z = h/2$, then regions with $z \geq z_{y2}$ experience tensile strain. Substituting $z_y = z_{y2}$ in the first of Eq. (17), the critical axial load P_{cr} which brings $z \geq z_{y1}$ up to the yield is:

$$P_{cr} = b\frac{S_y(z_2)}{E(z_2)}\frac{B_0^{ep}B_2^{ep} - (B_1^{ep})^2}{B_2^{ep} - B_1^{ep}z_2} + b\frac{(A_0' - A_0)(B_2^{ep} - B_1^{ep}z_2) + (A_1' - A_1)(-B_1^{ep} + B_0^{ep}z_2)}{B_2^{ep} - B_1^{ep}z_2} \tag{38}$$

When points above the $z_2 = z_{y2}$ are in plastic region and the other side of the beam has not begun to yield, then: $z_{y1} = -h/2$ and Eq. (32) results that coefficients A_0 and A_1 be equal to zero. If the axial load increases until the beam cross section at $z_1 \leq z_{y1}$ yields, then using the second of Eq. (17) results the value of z_{y2} corresponding to the z_{y1} as follow:

$$z_{y2} = \frac{B_2^{ep}}{B_1^{ep}} - (\frac{B_2^{ep}}{B_1^{ep}} - z_{y1})\frac{N_3}{N_4} \tag{39}$$

Wherein, N_3 and N_4 are:

$$\begin{aligned} N_3 = {} & \frac{S_y(z_{y2})}{E(z_{y2})}(B_0^{ep}B_2^{ep} - (B_1^{ep})^2) + (A_0' - A_0)(B_2^{ep} - B_1^{ep}z_{y2}) \\ & + (A_1' - A_1)(-B_1^{ep} + B_0^{ep}z_{y2}) \end{aligned} \tag{40}$$

$$\begin{aligned} N_4 = {} & -\frac{S_y(z_{y1})}{E(z_{y1})}(B_0^{ep}B_2^{ep} - (B_1^{ep})^2) + (A_0' - A_0)(B_2^{ep} - B_1^{ep}z_{y1}) \\ & + (A_1' - A_1)(-B_1^{ep} + B_0^{ep}z_{y1}) \end{aligned} \tag{41}$$

Since N_1 through N_4 are nonlinear functions of z_{y1} and z_{y2}, generally analytical solution for Eqs. (35) and (39) is not available and they should be solved numerically.

4.2 *Elastic-Plastic Bending Load*

Now the beam is assumed under pure bending load which results to partial yielding of the beam cross section. The same as the case for pure axial load, four different conditions are considered. These conditions are: a) Yield begins form $z = -h/2$, but the edge $z = h/2$ has not be begun to yield. b) Yield begins from $z = -h/2$ and the edge $z = h/2$ has begun to yield. c) Yield begins from $z = h/2$, but the edge $z = -h/2$ has not begun to yield. d) Yield begins from $z = h/2$ and $z = -h/2$ has also begun to yield.

Due to the assumed direction of bending load shown in Fig. (1), for case (a) which $z \leq z_{y1}$ is yielded but $z = h/2$ is not yielded, the regions with $z \leq z_{y1}$, experience compressive strain. Assuming $z_y = z_{y1}$ and $z_{y2} = 0$ in second of the Eq. (17), the critical bending moment M_{cr} which brings the regions with $z \leq z_{y1}$ up to yield is:

$$\begin{aligned} M_{cr} = {} & -b\frac{S_y(z_{y1})}{E(z_{y1})}\frac{B_0^{ep}B_2^{ep} - (B_1^{ep})^2}{-B_1^{ep} + B_0^{ep}z_{y1}} \\ & + b\frac{(-A_0' + A_0)(-B_2^{ep} + B_1^{ep}z_{y1}) + (-A_1' + A_1)(B_1^{ep} - B_0^{ep}z_{y1})}{-B_1^{ep} + B_0^{ep}zy1} \end{aligned} \tag{42}$$

If the bending moment is increased such that $z = h/2$ also begins to yield up to the $z = z_{y2}$ as described for case (b), then Eq. (42) is applicable to obtain the critical bending moment which brings the regions $z \geq z_{y2}$) up to the yield. But z_{y1} corresponding to z_{y2} have to be obtained from the first of Eq. (17) as follows:

$$z_{y1} = \frac{-B_1^{ep} + B_0^{ep} z_{y2}}{-B_1^{ep} + B_0^{ep} z_{y1}} - \frac{N_5}{N_6} \tag{43}$$

Wherein, N_5 and N_6 are:

$$\begin{aligned} N_5 = {} & \frac{S_y(z_{y2})}{E(z_{y2})}(B_0^{ep}B_2^{ep} - (B_1^{ep})^2) + (-A_0' + A_0)(-B_2^{ep} + B_1^{ep} z_{y2}) \\ & + (-A_1' + A_1)(B_1^{ep} - B_0^{ep} z_{y2}) \end{aligned} \tag{44}$$

$$\begin{aligned} N_6 = {} & -\frac{S_y(z_{y1})}{E(z_{y1})}(B_0^{ep}B_2^{ep} - (B_1^{ep})^2) + (-A_0' + A_0)(-B_2^{ep} + B_1^{ep} z_{y1}) \\ & + (-A_1' + A_1)(B_1^{ep} - B_0^{ep} z_{y1}) \end{aligned} \tag{45}$$

For the case (c) which $z \geq z_{y2}$ is in yield condition but $z = -h/2$ is not yielded, the region with $z \geq z_{y2}$ experiences compressive strain. Assuming $z_y = z_{y2}$ and $z_{y1} = 0$ in the second of Eq. (17), the critical bending moment M_{cr} which brings the regions with $z \geq z_{y2}$ up to the yield is:

$$\begin{aligned} M_{cr} = {} & -b\frac{S_y(z_{y2})}{E(z_{y2})}\frac{B_0^{ep}B_2^{ep} - (B_1^{ep})^2}{-B_1^{ep} + B_0^{ep} z_{y2}} \\ & + b\frac{(A_0' - A_0)(-B_2^{ep} + B_1^{ep} z_{y2}) + (A_1' - A_1)(B_1^{ep} - B_0^{ep} z_{y2})}{-B_1^{ep} + B_0^{ep} zy2} \end{aligned} \tag{46}$$

If the bending moment is increased such that $z = -h/2$ also begins to yield up to the $z = z_{y1}$ (i.e as described for case (d)), then Eq. (46) is applicable to obtain the critical bending moment which brings the regions $z \leq z_{y1}$) up to the yield. But z_{y2} corresponding to z_{y1} have to be estimated from the first of Eq. (17). That is:

$$z_{y1} = \frac{-B_1^{ep} + B_0^{ep} z_{y1}}{-B_1^{ep} + B_0^{ep} z_{y2}} - \frac{N_7}{N_8} \tag{47}$$

Wherein, N_7 and N_8 are:

$$N_7 = \frac{S_y(z_{y1})}{E(z_{y1})}(B_0^{ep}B_2^{ep} - (B_1^{ep})^2) + (A_0' - A_0)(-B_2^{ep} + B_1^{ep} z_{y1}) + (A_1' - A_1)(B_1^{ep} - B_0^{ep} z_{y1}) \tag{48}$$

$$N_8 = -\frac{S_y(z_{y2})}{E(z_{y2})}(B_0^{ep}B_2^{ep} - (B_1^{ep})^2) + (A_0' - A_0)(-B_2^{ep} + B_1^{ep} z_{y2}) + (A_1' - A_1)(B_1^{ep} - B_0^{ep} z_{y2}) \tag{49}$$

4.3 Elastic-Plastic Axial and Bending Load

In this case it is assumed that an axial load P less than axial collapse load P_{col} is imposed to the beam, then the critical bending moment M_{cr} which brings specified regions of the cross section up to the yield is calculated. To this aim the same four different cases which is considered in previous load case is considered. For the case (a), described in previous section which $z \leq z_{y1}$

is yielded and experiences compressive strain, the critical bending moment M_{cr} which brings $z \leq z_{y1}$ up to the yield is:

$$\begin{aligned} M_{cr} = &-b\frac{S_y(z_{y1})}{E(z_{y1})}\frac{B_0^{ep}B_2^{ep}-(B_1^{ep})^2}{-B_1^{ep}+B_0^{ep}z_{y1}} \\ &+b\frac{(P/b-A_0'+A_0)(-B_2^{ep}+B_1^{ep}z_{y1})+(-A_1'+A_1)(B_1^{ep}-B_0^{ep}z_{y1})}{-B_1^{ep}+B_0^{ep}zy1} \end{aligned} \tag{50}$$

For case (b) of the previous section which $z \leq z_{y1}$ is yielded and experiences compressive strains and at the same time $z \geq z_{y2}$ is yielded, the critical bending moment M_{cr} with combination of axial load P, which brings the specified section of the beam up to the yield is calculated from Eq. (50). Value of z_{y1} corresponding to specified z_{y2} have to be calculated from the following equation:

$$z_{y1} = \frac{-B_1^{ep}+B_0^{ep}z_{y2}}{-B_1^{ep}+B_0^{ep}z_{y1}} - \frac{N_5}{N_6} \tag{51}$$

Wherein, N_9 and N_{10} are:

$$\begin{aligned} N_9 = &\frac{S_y(z_{y2})}{E(z_{y2})}(B_0^{ep}B_2^{ep}-(B_1^{ep})^2)+(P/b-A_0'+A_0)(-B_2^{ep}+B_1^{ep}z_{y2}) \\ &+(-A_1'+A_1)(B_1^{ep}-B_0^{ep}z_{y2}) \end{aligned} \tag{52}$$

$$\begin{aligned} N_{10} = &-\frac{S_y(z_{y1})}{E(z_{y1})}(B_0^{ep}B_2^{ep}-(B_1^{ep})^2)+(P/b-A_0'+A_0)(-B_2^{ep}+B_1^{ep}z_{y1}) \\ &+(-A_1'+A_1)(B_1^{ep}-B_0^{ep}z_{y1}) \end{aligned} \tag{53}$$

In case (c) which $z \geq z_{y2}$ is yielded and the corresponding strain is tensile, the critical bending moment with combination of axial load which brings $z \geq z_{y2}$ up to the yield is:

$$\begin{aligned} M_{cr} = &-b\frac{S_y(z_{y2})}{E(z_{y2})}\frac{B_0^{ep}B_2^{ep}-(B_1^{ep})^2}{-B_1^{ep}+B_0^{ep}z_{y2}} \\ &+b\frac{(P/b+A_0'-A_0)(-B_2^{ep}+B_1^{ep}z_{y2})+(A_1'-A_1)(B_1^{ep}-B_0^{ep}z_{y2})}{-B_1^{ep}+B_0^{ep}zy2} \end{aligned} \tag{54}$$

and finally for the last case which $z \geq z_{y2}$ is yielded and strain is tensile, while the regions $z \leq z_{y1}$ are also yielded, the value of z_{1y} corresponding to z_{2y} is calculated from the following equation.

$$z_{y1} = \frac{-B_1^{ep}+B_0^{ep}z_{y1}}{-B_1^{ep}+B_0^{ep}z_{y2}} - \frac{N_{11}}{N_{12}} \tag{55}$$

Wherein, N_{11} and N_{12} are:

$$\begin{aligned} N_{11} = &\frac{S_y(z_{y1})}{E(z_{y1})}(B_0^{ep}B_2^{ep}-(B_1^{ep})^2)+(P/b+A_0'-A_0)(-B_2^{ep}+B_1^{ep}z_{y1}) \\ &+(A_1'-A_1)(B_1^{ep}-B_0^{ep}z_{y1}) \end{aligned} \tag{56}$$

$$\begin{aligned} N_{12} = &-\frac{S_y(z_{y2})}{E(z_{y2})}(B_0^{ep}B_2^{ep}-(B_1^{ep})^2)+(P/b+A_0'-A_0)(-B_2^{ep}+B_1^{ep}z_{y2}) \\ &+(A_1'-A_1)(B_1^{ep}-B_0^{ep}z_{y2}) \end{aligned} \tag{57}$$

5 Collapse load

In collapse condition, the cross section of the beam is fully plastic. Since in this case there is not any elastic region across the section of the beam, the conditions (7) and (8) are independent of the values c_0 and c_1. So the constants of integration remain undefined. As the results the strain distribution is undetermined which is in coherence with collapse condition of the beam. Using boundary conditions, one may find the collapse load of a beam and the point z_1 which the direction of stress is changed. The same as previous sections, three different load cases are considered in this section.

5.1 Case 1: $P \neq 0\,,\, M = 0$

In this case the collapse axial load is obtained in absence of the bending moment. Since the material property of the beam is nonhomogeneous, although the beam experiences tensile load, but in some parts of the cross section, it experiences compressive stress in order to satisfy the condition $M = 0$. Substituting Eq. (5) into boundary condition (7):

$$\int_{-h/2}^{z_1} -S_y dz + \int_{z_1}^{h/2} S_y dz = P/b \tag{58}$$

As the results, the collapse axial load is obtained due to the following equation:

$$P_{col} = b\{2z_1 S_{ym} + (S_{yc} - S_{ym})\frac{h}{n+1}[2(z_1/h + 1/2)^{n+1} - 1]\} \tag{59}$$

where z_1 is the height of the point which axial stress changes its direction from tension to compression. Now, the second boundary condition given in Eq. (8) with $M = 0$ is used to evaluate z_1:

$$\int_{-h/2}^{z_1} -S_y z dz + \int_{z_1}^{h/2} S_y z dz = 0 \tag{60}$$

Substituting Eq. (2) into Eq. (60) results:

$$2\frac{(z_1/h + 1/2)^{n+2}}{n+2} - \frac{(z_1/h + 1/2)^{n+1}}{n+1} + \frac{(z_1^2 - h^2/4)S_{ym}}{h^2(S_{yc} - S_{ym})} - \frac{1}{n+2} + \frac{1}{2(n+1)} = 0 \tag{61}$$

Using any numerical iterative method, one may find the solution of Eq. (61) for z_1. To obtain the collapse axial load, it is assumed that yield begins from $z = -h/2$, however there is not any difference to result of collapse axial load when yield begins from $z = h/2$.

5.2 Case 2: $P = 0\,,\, M \neq 0$

In this case the beam is under pure bending load. Substituting Eq. (5) into boundary condition (8):

$$\int_{-h/2}^{z_1} -S_y z dz + \int_{z_1}^{h/2} S_y z dz = M/b \tag{62}$$

followed by substitution of Eq. (2) into above equation, results to:

$$\begin{aligned} M_{col} = b\{&-S_{ym} z_1^2 + S_{ym}\frac{h^2}{4} - 2(S_{yc} - S_{ym})h^2\frac{(z_1/h + 1/2)^{n+2}}{n+2} \\ &+ (S_{yc} - S_{ym})h^2\frac{(z_1/h + 1/2)^{n+1}}{n+1} + (S_{yc} - S_{ym})h^2[\frac{1}{n+2} - \frac{1}{2(n+1)}]\} \end{aligned} \tag{63}$$

Table 1 FG beam material properties

Material property	Metal	Ceramic
Modulus of elasticity	106 (GPa)	131.4 (GPa)
Tensile yield strength	99.7(MPa)	145.6 (MPa)

In order to obtain the height of the point, which stress changes its direction, the boundary condition (7) is considered. Since the axial load is zero, the boundary condition (7) results to:

$$\int_{-h/2}^{z_1} -S_y dz + \int_{z_1}^{h/2} S_y dz = 0 \tag{64}$$

Substituting Eq. (2) into above equation, the required condition to obtain the neutral axis is found. That is:

$$2S_{ym}z_1 + 2h(S_{yc} - S_{ym})\frac{(z_1/h + 1/2)^{n+1}}{n+1} - h\frac{S_{yc} - S_{ym}}{n+1} = 0 \tag{65}$$

5.3 *Case 3:* $P \neq 0$, $M \neq 0$

Finally in the last case an axil load less than the collapse load is applied to the beam. Then a bending moment M_{col} is applied to the beam to make the cross section of the beam fully plastic. To evaluate the collapse bending load, two different condition which yield begins from top or bottom surface of the beam is considered. Since the terms in boundary condition (8) are not changed in both cases, Eq. (63) is applicable for both cases. The only difference is that the combination of axial load and bending load results to different value of z_1 with respect to previous one. If yield begins from $z = -h/2$, then using the condition (7), z_1 is found as:

$$2S_{ym}z_1 + 2h(S_{yc} - S_{ym})\frac{(z_1/h + 1/2)^{n+1}}{n+1} - h\frac{S_{yc} - S_{ym}}{n+1} = P/b \tag{66}$$

and when yield begins from $z = h/2$, z_1 is:

$$-2S_{ym}z_1 - 2h(S_{yc} - S_{ym})\frac{(z_1/h + 1/2)^{n+1}}{n+1} + h\frac{S_{yc} - S_{ym}}{n+1} = P/b \tag{67}$$

To check that which of the equations (66) and (67) should be used, Eq. (63) is solved for both values of these two equations and the collapse bending load is considered as the minimum of these values.

6 Results and Discussion

In this section using the effective material properties given in Table (1), the yield, elastic-plastic and collapse behavior of the FG beam due to the imposed axial load, bending load and their combination are obtained. Setting $n = 0$ in equations (1) and (2), the FG beam changes to an isotropic beam with material properties identical to ceramic part of the beam. So, if in equations represented in previous sections, the value of n is set to zero, then they have to predict the behavior of the beam the same as an isotropic one with the material properties given for ceramic part of the FG beam.

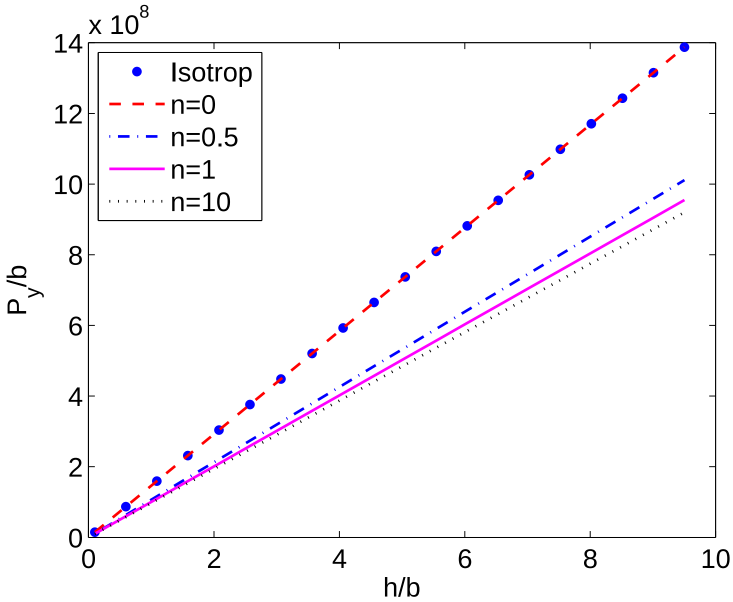

Figure 2 P_y/b versus h/b for different values of n

6.1 Results for Pure Axial Load

In first example consider the beam to be subjected to pure axial load. The plot of yield axial load P_y per unit width of the beam b versus the normalized height h/b is shown in Fig. (2) for $n = 0, 0.5, 1$ and $n = 10$ using equation (18). On the other hand, for an isotropic beam the yield axial load is obtained from the following simple equation:

$$P_y = S_y bh \tag{68}$$

In Fig. (2), the plot of normalized yield axial load versus normalized height for $n = 0$ is compared for that of isotropic beam. As it can be seen from this figure, the results of FG beam for $n = 0$ are identical with the isotropic ceramic beam. On the other hand, this plot shows that yield axial load decreases with increasing the values of n. For the beam with dimensions $b = 0.01m$ and $h = 0.01m$ and $n = 0.5$, the yield axial load due to Eq. (18) is $10.638\ kN$ and the stress distribution for this case is plotted in Fig. (3) using Eq. (4). These stresses are compared with yield stress at cross section of the FG beam. If the load is increased up to the values that the FG beam collapses, then using Eq. (59), the normalized collapse load versus normalized height of the beam is obtained from Eq. (59), as shown in Fig. (4). On the other hand, since the stress distribution for isotropic beam is uniform, the collapse load and yield load are equal. So, Eq. (68) also shows the collapse axial load of an isotropic beam. It can be seen from Fig. (4), that the collapse load for the isotropic beam is identical to the FG beam with $n = 0$. Also this figure shows that increasing the values of n results to decreasing the collapse axial load of the FG beam. For a beam with dimensions $b = 0.01m$ and $h = 0.01m$ and $n = 0.5$, the collapse axial load due to Eq. (59) is $298.27\ kN$ and the stress distribution is plotted in Fig. (5). This figure shows that the cross section of the beam below $z_1 = 0.0048\ m$ experiences tensile yield stress while the points above $z_1 = 0.0048\ m$ experience compressive yield stress. It is interesting since for the applied tensile load, a region with compressive stress is produced. Now consider that the critical axial load P_{cr} which is between $P_y \leq P_{cr} \leq P_{col}$. Then the cross section of the beam begins to yield form one edge of the beam cross section and increasing the load will results to the fully plastic cross section. Figure (6) shows variation

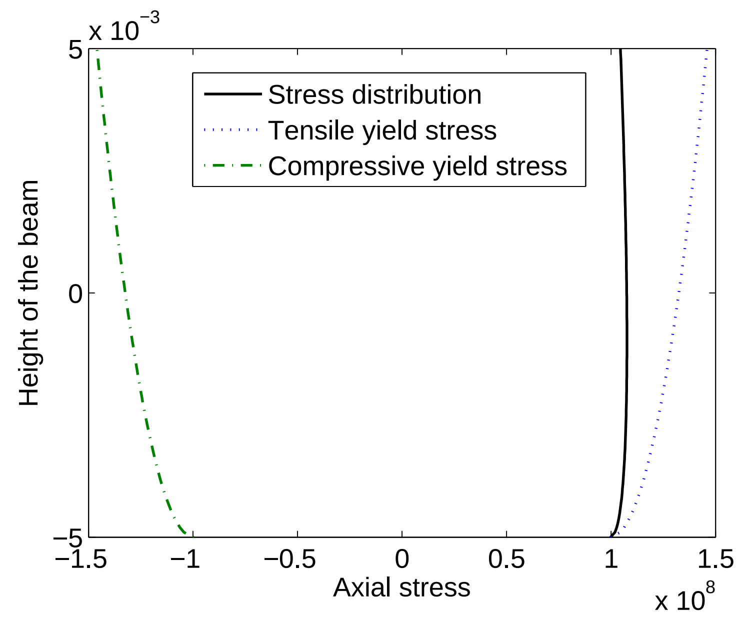

Figure 3 Stress distribution of the beam due to the yield axial load

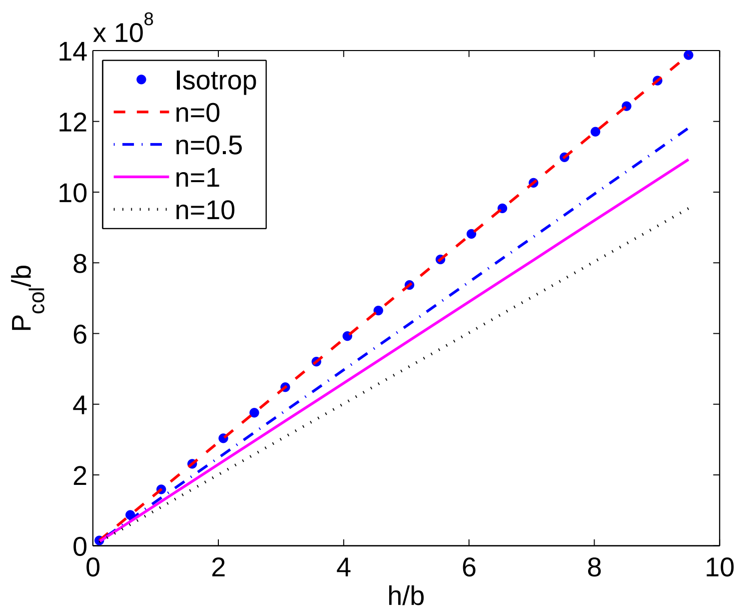

Figure 4 P_{col}/b versus h/b for different values of n

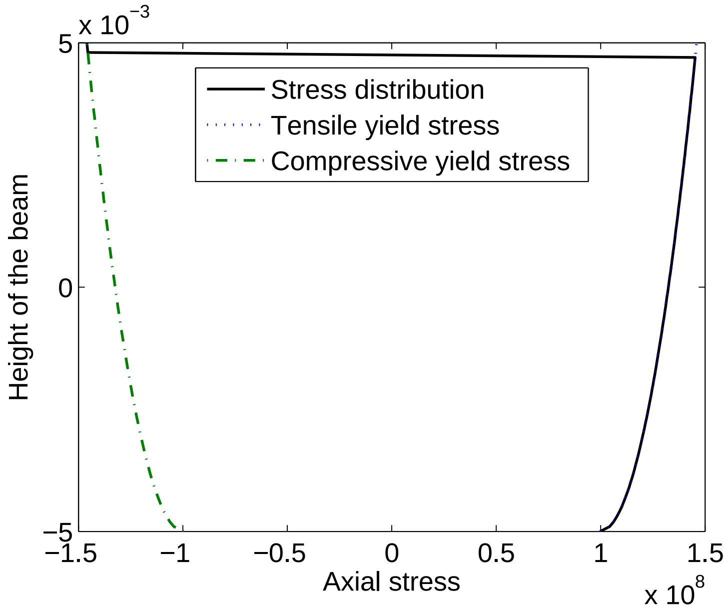

Figure 5 Stress distribution of the beam due to the collapse axial load

of $z_{y1/h}$ with respect to P_{cr}/b. The figure shows that when $n = 0$, the results are identical to those of isotropic beam. On the other hand, it shows that increasing the value of n, results to decrease in critical yield load. Solving Eq. (35) numerically, z_{y1} corresponding to beginning of yield at $z = h/2$ (i.e. $z_{y2} = h/2$) is: $z_{y1} = 0.0046\ m$, For a beam with previous given cross section properties. Substitution of these values in Eq. (34) results, $P_{cr} = 12.421\ kN$. The stress distribution corresponding to this load case is plotted in Fig. (7). As it can be seen from the figure, the region $z \leq 0.0046\ m$ is yielded and stress in this region is tensile. The region between $0.0046\ m$ and $0.005\ m$ is elastic. The point $z = h/2$ has begun to yield and stress at this point is compressive. Increasing the axial load more than $P_{cr} = 12.421\ kN$ results to development of compressive stress in cross section of the beam as described in previous example.

6.2 Results for Pure Bending Load

Here the results due to the imposed pure bending load is discussed. Figure (8) shows the plot of normalized yield bending moment M_y/b versus normalized height h/b of the FG and isotropic beams. For the isotropic beam, the yield bending moment is simply calculated as follow:

$$M_y = S_y \frac{bh^2}{6} \tag{69}$$

Figure (8) shows that the results of Eq. (19) for FG beam with $n = 0$ are identical with isotropic beam and the yield bending moment of FG beam is lower than isotropic ceramic beam. For the FG beam with dimensions $b = 0.01$, $h = 0.01$ and $n = 0.5$ the yield bending moment using Eq. (19) is $18.54\ Nm$. The stress distribution corresponding to $M_y = 18.54\ Nm$ is shown in Fig. (9). As it can be seen from this figure the stress at the point $z = -h/2$ is equal to yield stress at metal side of the FG beam and is compressive, assuming the bending moment is applied in positive direction. Using the Eq. (63), the normalized collapse bending moment M_{col}/b versus the normalized hight of the beam is plotted in Fig. (10) for the FG beam. On the other hand the

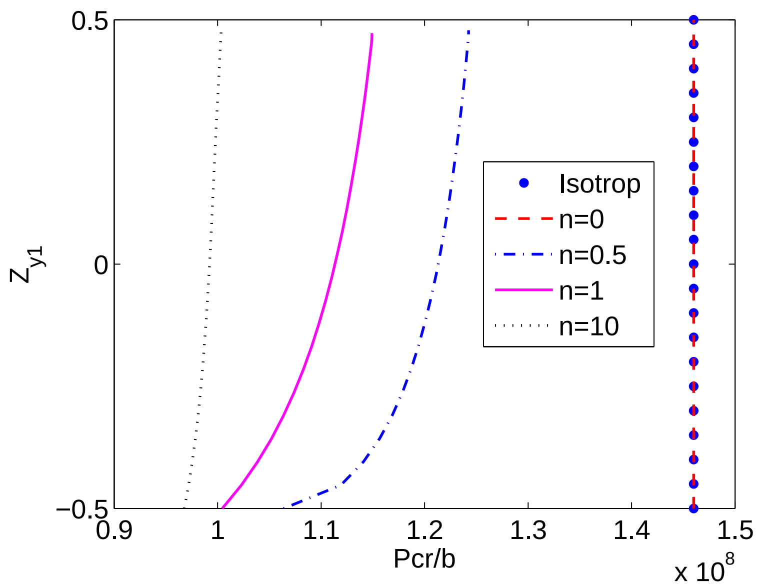

Figure 6 Variation of height of plastic region with respect to critical axial load

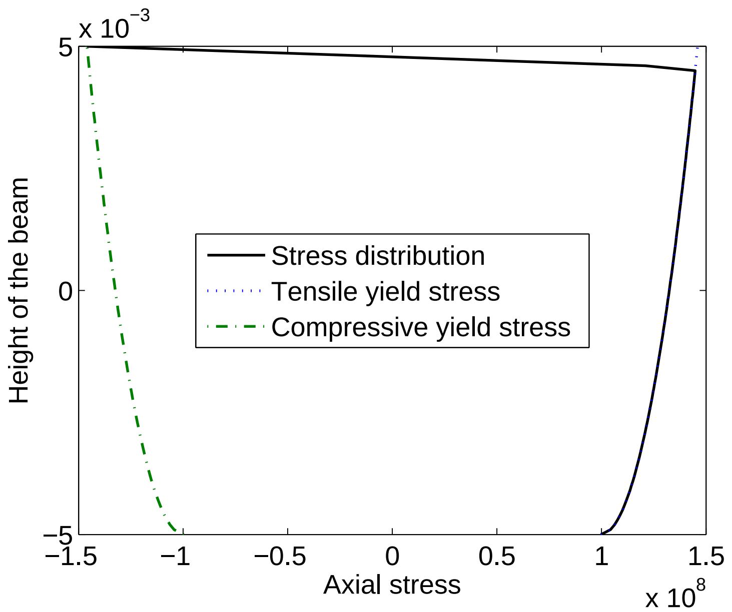

Figure 7 Stress distribution of the beam due to the elastic plastic axial load

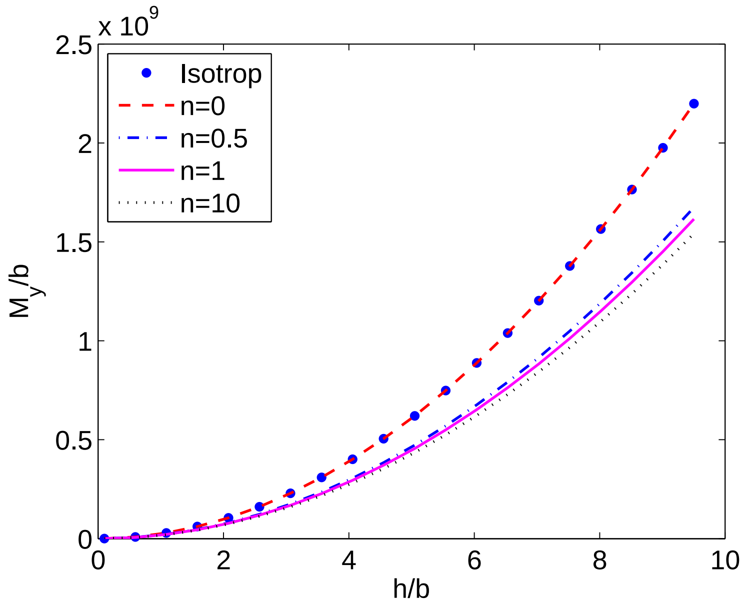

Figure 8 M_{col}/b versus h/b for different values of n

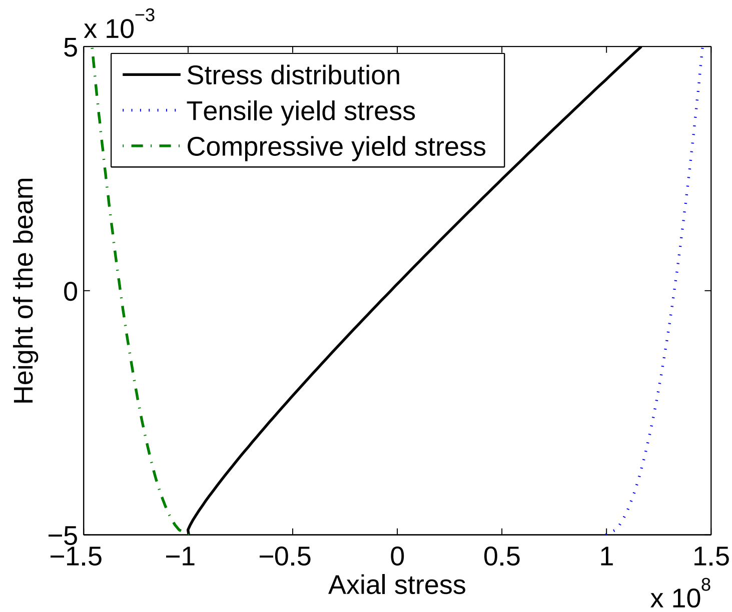

Figure 9 Stress distribution of the beam due to the yield bending moment

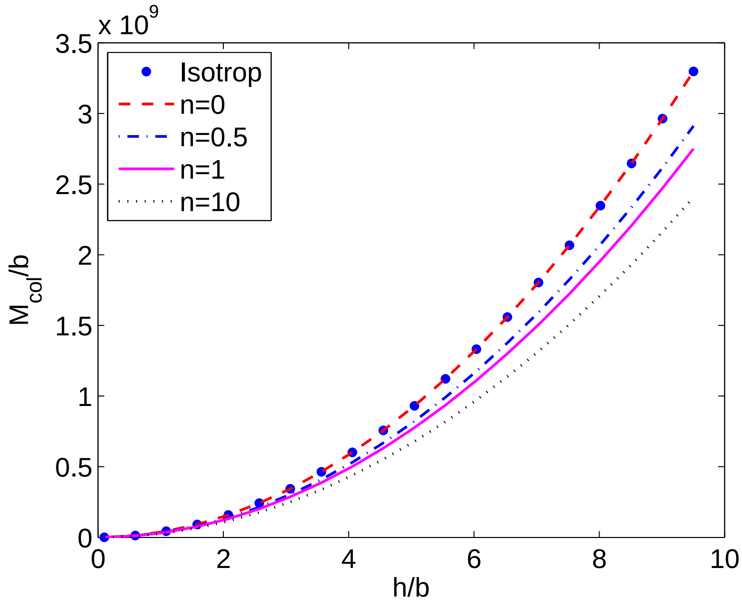

Figure 10 M_{col}/b versus h/b for different values of n

collapse bending moment for an isotropic beam simply is:

$$M_{col} = S_y \frac{bh^2}{4} \tag{70}$$

Fig. (10), shows that the collapse bending moment of the FG beam with $n = 0$ is identical with the collapse bending moment of the isotropic beam. This figure shows that increasing the value of n results to decrease in collapse bending moment. The collapse bending moment of the beam with previously specified dimensions, is $32.223\ Nm$. Fig. (11) is plot of stress distribution for this beam subjected to the collapse bending moment. The height of the beam which the direction of stress is changed according to numerical solution of Eq. (65) is $z_1 = 0.0004\ m$. The figure shows that the region bellow this point experiences compressive yield stress and the region above this point experiences tensile yield stress. If the bending moment imposed to the beam be less than the collapse moment and more than the yield moment, then the cross section of the beam flows partly into plastic region. In Fig. (12) the height of the yield point z_{y2} is plotted against the corresponding critical bending moment M_{cr} for a beam made of FG material. The critical bending moment of a beam made of isotropic materials which brings the beam cross section located above the z_{y2} up to the yield is:

$$M_{cr} = bS_y\left(\frac{h^2}{4} - \frac{z_{y2}^2}{3}\right) \tag{71}$$

Fig. (12) shows that the critical load of a beam made of FG material with $n = 0$ is identical to isotropic ceramic beam. Fig. (13) shows the stress distribution of the beam made of FG material with dimensions $h = 0.01\ m$, $b = 0.01\ m$ and $n = 0.5$ for the case which the top surface of the beam has begun to yield. The z_{y1} corresponding to this plastic region according to Eq. (43) is $-0.0042\ m$ and using the Eq. (42), the bending moment corresponding to $z_{y1} = -0.0042\ m$ and $z_{y2} = 0.005$ is $M_{cr} = 22.84\ Nm$.

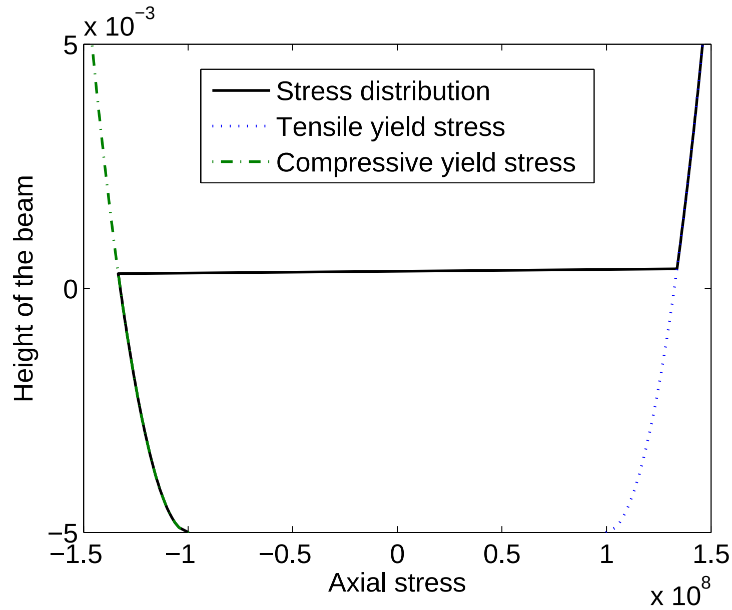

Figure 11 Stress distribution of the beam due to the collapse bending moment

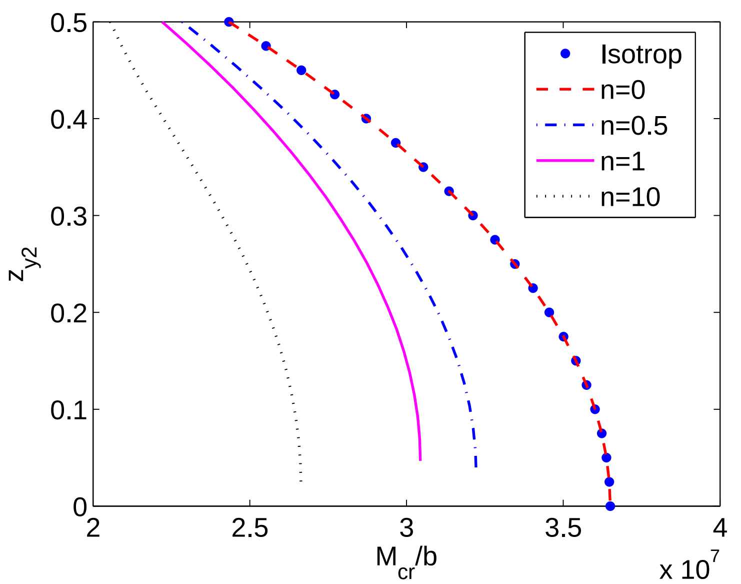

Figure 12 Variation of height of plastic region with respect to critical bending moment

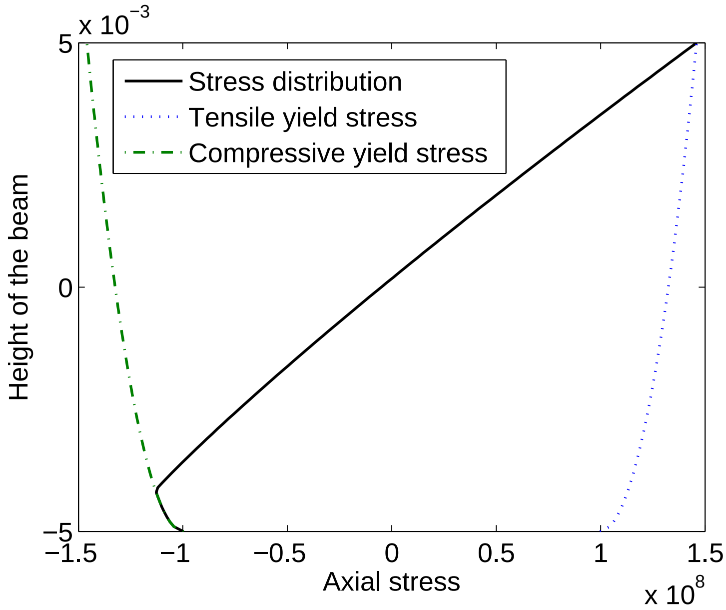

Figure 13 Stress distribution of the beam due to the elastic plastic bending moment

6.3 Results for Combination of Axial and Bending Load

In this section behavior of the beam made of FG materials subjected to combination of axial and bending load is investigated. For an isotropic beam the yield bending moment M_y, which brings the cross section of a beam subjected to the axial load P up to the yield, is:

$$M_y = \frac{bh^2}{6} S_y - P\frac{h}{6} \tag{72}$$

Figure (14), shows the interaction diagram between M_y and P. This figure is plotted using results of Eq. (50) through (57). The figure shows that the interaction diagram for isotropic ceramic beam is identical to the FG beam with $n = 0$. Also, it shows that increasing the value of n, results to decrease in ability of the beam to support the combination of bending and axial load. Figure (15) shows the yield interaction diagram for the case where the location of the ceramic and metal is replaced. As it can be seen from Fig. (15), increasing the value of n in this case results to increase in ability of the beam to support the combination of bending and axial load. In (16) and (17), the collapse interaction diagram is obtained for the same conditions of Figs. (14) and (15). The collapse bending moment M_{col} for an isotropic beam subjected to axial load P, is:

$$M_{col} = bS_y(-z_1^2 + \frac{h^2}{4}) \tag{73}$$

Wherein:

$$z_1 = -\frac{P}{2bS_y} \tag{74}$$

These figures show that the collapse interaction diagram of FG beam with $n = 0$ is identical to collapse interaction diagram of the isotropic beam. Also, for the case where metallic part of the FG beam is located at $z = -h/2$, increasing the value of n_1 results to decrease the ability of the beam to support the combination of axial and bending loads. Replacing the location of the metallic and ceramic parts of the FG beam, results to increase in ability of the beam to support the combination of axial and bending load, as the value of n is increased.

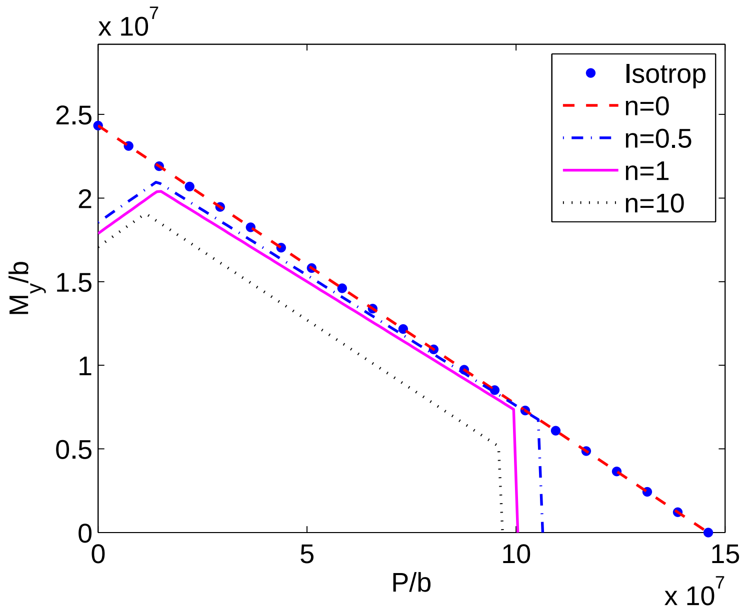

Figure 14 Yield interaction diagram for combination of axial and bending moment

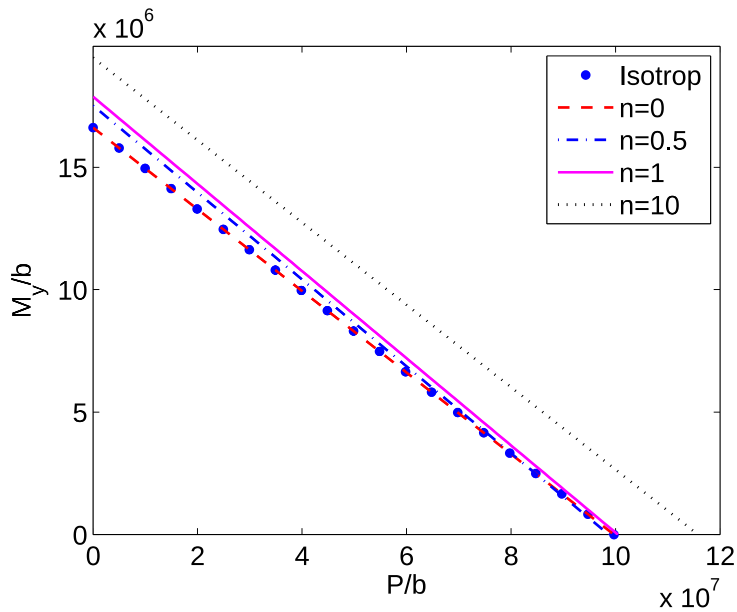

Figure 15 Effect of changing the place of metal with ceramic on yield interaction diagram

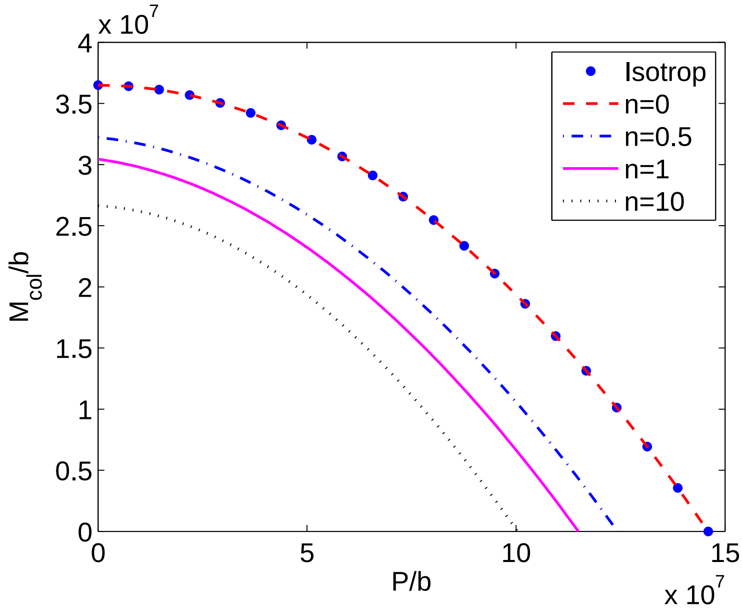

Figure 16 Collapse interaction diagram for combination of axial and bending moment

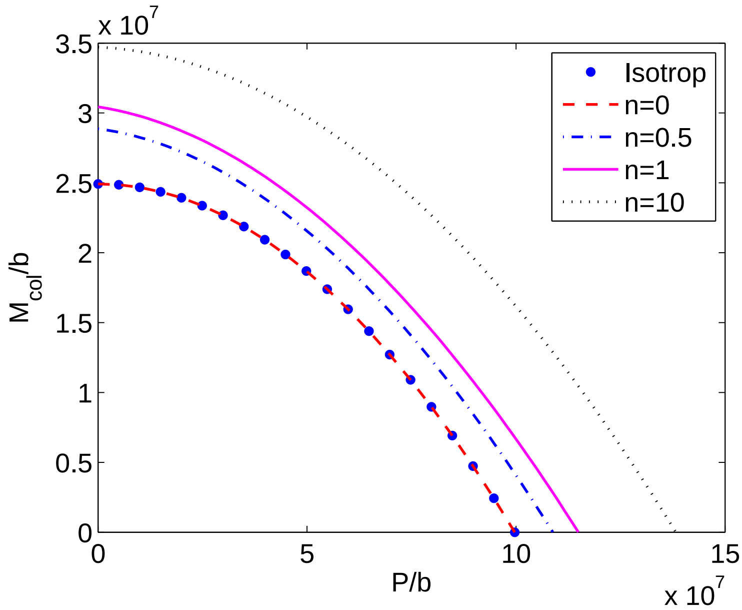

Figure 17 Effect of changing the place of metal with ceramic on collapse interaction diagram

7 Conclusion

In this paper the required equations to obtain the yield axial load, yield bending load and the combination of axial and bending loads which bring the cross section of the beam up to the yield, are obtained. Also the equations required to obtain the elastic-plastic and collapse behavior of the beam are obtained. Curves plotted in this paper show that predicted results by these equations for an FG beam with $n = 0$ are identical with those of the isotropic beam. It is concluded that for the case where the metallic part of the beam is located at $z = -h/2$ the values of yield, elastic-plastic and collapse loads are less than the isotropic beam made of ceramic. On the other hand, the interaction diagrams between bending moment and axial load for both of the yield and collapse conditions show that: increase in values of n results to decrease in ability of the structure to withstand the imposed loads when the metallic part is located at $z = -h/2$. Locating the metallic part of the beam at $z = h/2$ enhances the ability of the structure to withstand the imposed loads with increasing the value of n.

References

[1] Lee, Y.D., and Erdogan, F., ”Residual/Thermal Stress in FGM and Laminated Thermal Barrier Coating”, International Journal of Fracture, pp. 145-165, Vol. 69, No. 2, (1994).

[2] Suresh, S., and Mortensen, A., ”Functionally Graded Metals and Metal-Ceramic Composites: Part 2 Thermomechanical Behavior”, International Materials Review, pp. 85-116, Vol. 42., No. 3, (1997).

[3] Garino, J.P., ”Modern Ceramic-on-Ceramic Total Hip Systems in the United States: Early Results”, Clinical Orthopaedics And Related Research, pp. 41-47, (2000).

[4] Fujihara, K., Teo, K., Gopal, R., Loh, P.L., Ganesh, V.K., Ramakrishna, S., ”Fibrous Composite Materials in Dentistry and Orthopedics: Review and Applications” Composite Science Technology, pp. 775-788, Vol. 64, No. 6, (2004).

[5] Eslami, M.R., Babaei, M.H., and Poultangari R., ”Thermal and Mechanical Stresses in a Functionally Graded Thick sphere”, International Journal of Pressure Vessels and Piping, pp. 522-527, Vol. 82, No. 7, (2005).

[6] Bahtui, A., and Eslami, M.R., ”Coupled Thermoelasticity of Functionally Graded Cylindrical Shells”, Mechanics Research Communications, pp. 1-18, Vol. 34, No. 1, (2007).

[7] Peng, X., Hu, N., Zheng, H. and Fang, C., ”Analysis of shakedown of FG Bree Plate Subjected to Coupled Thermal-Mechanical Loadings”, Acta Mechanica Solida Sinica, pp. 95-108, Vol. 22, No. 2, (2009).

[8] Peng, X, Zhang, H., Hu, N. and Fang, C., ”Static and Kinematic Shakedown Analysis of FG Plate Subjected to Constant Mechanical Load and Cyclically Varying Temperature Change”, Composite Structures, pp. 212-221, Vol. 91, No. 2, (2009).

[9] Nemat-Alla, M., Ahmed, K.I.E., Hassab-Allah, I., ”Elastic-Plastic Analysis of Two-Dimensional Functionally Graded Materials under Thermal Loading”, International Journal of Solids and Structures, pp. 2774-2786, Vol. 46, No. 14-15, (2009).

[10] Reddy, J.N., Chin, C.D., ”Thermomechanical Analysis of Functionally Graded Cylinders and Plates”, Journal of Thermal Stresses, Vol. 21, No. 6, pp. 593-626, (1998).

Nomenclature

b:	Width of the beam
h:	Height of the beam
z_1:	Height of the point which stress changes its direction in collapse condition
z_n:	Height of the neutral axis
z_{y1}:	Height of the underneath plastic region
z_{y2}:	Height of the upper plastic region
z_n^e:	Height of the neutral axis at beginning of the yield
z_n^{ep}:	Height of neutral axis in elastic-plastic region
E_c:	Modulus of elasticity of ceramic
E_m:	Modulus of elasticity of metal
M:	Bending moment
M_{col}:	Collapse bending moment
M_{cr}:	Critical bending moment in plastic region
M_y:	Yield Bending moment
P:	Axial load
M_{col}:	Collapse axial load
P_{cr}:	Critical axial load in plastic region
P_y:	Yield axial load
S_{yc}:	Yield strength of the ceramic
S_{ym}:	Yield strength of the metal

Greek symbols

ϵ_x:	Axial total strain
ϵ_x^e:	Axial elastic strain
ϵ_x^p:	Axial elastic strain
σ_x:	Axial stress

چکیده

در این تحقیق رفتار الاستیک-پلاستیک یک تیر ساخته شده از مواد تابعی مورد بررسی قرار گرفته است. تیر تحت بارهای ثابت محوری و خمشی قرار گرفته و مقادیر بحرانی این بارها در شرایط تسلیم، فروپاشی و الاستیک-پلاستیک تعیین شده است. تغییرات مدول الاستیسیته و مقاومت تسلیم در ضخامت تیر با استفاده از رابطه نمایی نشان داده شده است. از منحنی پلاستیک کامل برای مدل کردن رفتار پلاستیک تیر استفاده شده است. منحنی‌های اثر متقابل بین نیروی محوری و ممان خمشی برای هر دو حالت تسلیم و فروپاشی ترسیم شده است. اثر تابع نمائی بر شرایط تسلیم و فروپاشی تیر بررسی شده است. نتایج تیر تابعی، به تیر هموژن ساده شده و صحت آنها با مقایسه با نتایج موجود در سابقه علمی، بررسی شده است.