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A Differential Transform Approach for Modal Analysis of Variable Thickness Two-directional FGM Circular Plates on Elastic Foundations

Based on the differential transformation method, a semi-analytical solution is developed for free vibration and modal stress analyses of variable thickness two-directional functionally graded circular plates with restrained edges, resting on elastic foundations. Variations of the material and geometry parameters are monitored by five distinct exponential functions. The presented non-dimensional solution covers complex combinations of the material properties, edge conditions, and parameters of the elastic foundation. Results reveal that by a proper adoption of the distribution of the material properties, a somewhat uniform strength may be attained for the cross sections without the need to change the geometric parameters of the plate.

Keywords: differential transform, vibration, modal stress, circular plate, two-directional functionally graded materials, elastic foundation.

1 Introduction

Many engineering components may be modeled directly or through employing some simplifications, to circular plates with variable or constant thickness. Due to offering the capability to monitor the material properties to meet the strength, toughness, heat transfer, and other design requirements, plates made of functionally graded materials resting on simple, elastic, or complex foundations, have recently drawn attention of numerous researchers. The main advantage of using the functionally graded materials is the capability to accurately monitor variations of the material properties of the mixture of the constituent materials at the microstructure or continuum level. Therefore, e.g. one may achieve the strength requirements by means of increasing the material strength at the more critical sections instead of increasing the cross section at the mentioned regions. Two or three-directional functionally graded materials enable a more accurate monitoring of the material properties.

Free vibration and modal stress characteristics of the functionally graded plates, affect their transient dynamic responses significantly. Indeed, the transient response of the plate may be determined through a mode superposition technique. Majority of the modal analyses of the

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circular plates have been accomplished for the isotropic ones [1,2]. Gupta et al. [3] analyzed free axisymmetric vibrations of non-homogeneous isotropic circular plates with nonlinear thickness variations on the basis of the classical plate theory employing the differential quadrature method (DQM).

Some researchers have studied free vibration of functionally graded circular plates whose material properties vary in the transverse or radial directions according to a power law or exponential functions (one-directional functionally graded materials). Using the finite element procedure, Prakash and Ganapathi [4] investigated asymmetric free vibration characteristics of the functionally graded circular plates. Nie and Zhong [5] proposed a semi-analytical method which uses the state space method and the one-dimensional differential quadrature method, for three-dimensional free and forced vibration analyses of functionally graded circular plates with various boundary conditions. Allahverdizadeh et al. [6] developed a semi-analytical approach for nonlinear free and forced axisymmetric vibrations of a thin circular functionally graded plate, using a time averaging technique. Hosseini-Hashemi et al. [7] studied free vibration of radially FGM sectorial plates of variable thickness on elastic foundations, using the classical plate theory. Dong [8] investigated three-dimensional free vibration of functionally graded annular plates with different boundary conditions, using the Chebyshev–Ritz method. A free vibration analysis of thick functionally graded plates supported by two-parameter elastic foundations was presented by Malekzadeh [9], using the three-dimensional theory of elasticity. A semi-analytical approach composed of DQM and a series solution was adopted to solve the governing equations.

The differential transform method (DTM) is a semi-analytical technique the uses Taylor's series expansion. Employing DTM, it is possible to obtain highly convergent and accurate results and exact solutions for differential or integro-differential equations [10]. By using this method, the governing differential equations can be reduced to recurrence relations and the boundary conditions may be transformed into a set of algebraic equations. Some researchers have successfully used the DTM in solving the eigenvalue problems [11–13]. Yeh et al. [14] analyzed free vibration of the rectangular thin plates, using a hybrid method which combines the finite difference and the differential transformation methods. Yeh et al. [15] studied large deflections of the orthotropic rectangular thin plates, employing a similar hybrid method. Shin et al. [16] applied the generalized differential quadrature method and the DTM for vibration analysis of circular arches with variable cross-section. Yalcin [17] analyzed free vibration of thin circular plates with various boundary conditions, using the differential transform method.

In the present paper, a semi-analytical solution for free vibration and modal stress analyses of two-directional functionally graded circular plates with restrained edges, resting on two-parameter elastic foundations is developed based on the differential transformation method. Thickness of the plate may vary in the radial direction. Although it is known that the exact and analytical solutions are generally restricted to some specific geometry, boundary, or loading conditions, present solution may be used for complex combinations of the material properties, boundary conditions, and foundation stiffnesses. Plates with free, simply-supported, and clamped edge conditions resting on two-parameter foundations are considered in the present semi-analytical solution.

2 Description of the two-directional variations of the material properties

Consider a circular plate with an outer radius *b* made of a two-directional functionally graded material supported by an elastic foundation with Winkler's (normal) and Pasternak's (shear) coefficients. The external edge of the plate may experience free, simply-supported, or clamped axisymmetric boundary conditions. Variations of the material properties of the plate are considered to be exponential in both radial and transverse directions. It is assumed that the material properties at any arbitrary point of the thickness located at a distance z from the mid-plane of the plate, may be determined based on the following equations:

$$E(r,z)=E_0 e^{m_1(\frac{z}{h}+\frac{1}{2})+n_1\frac{r}{b}} \tag{1}$$

$$\rho(r,z)=\rho_0 e^{m_2(\frac{z}{h}+\frac{1}{2})+n_2\frac{r}{b}} \tag{2}$$

Therefore, each of Young's modulus E and mass density ρ functions is defined based on multiplication of two exponential functions with dimensionless exponents. E_0 and ρ_0 are the reference Young's modulus and mass density, respectively. The exponents m_1, m_2, n_1, and n_2 may be positive or negative. Variations of Poisson's ratio are negligibly small and therefore may be ignored [18-21]. Therefore, this coefficient may be considered to be constant all over.

Thickness of the plate may increase or decrease in the radial direction according to a prescribed function such as the following one:

$$h(r)=h_0 e^{n_3\frac{r}{b}} \tag{3}$$

Therefore, in Eqs. (1) and (2), h that is appeared in the exponents, itself is a function of the radial coordinate r:

$$E(r,z)=E_0 e^{m_1\left(\frac{z}{h_0 e^{n_3\frac{r}{b}}}+\frac{1}{2}\right)+n_1\frac{r}{b}} \quad , \qquad \rho(r,z)=\rho_0 e^{m_2\left(\frac{z}{h_0 e^{n_3\frac{r}{b}}}+\frac{1}{2}\right)+n_2\frac{r}{b}} \tag{4}$$

3 The governing equations

Usually, the classical plate theory leads to sufficiently accurate results for the thin plates [22]. Since is intended to derive governing equations suitable for modal or a free vibration analysis purpose, it is implicitly assumed that values of the externally applied forces (including the body forces) are zero. Furthermore, the rigid-body modes are of no concern in the present study. Therefore, since the material properties are dependent on the radial coordinate r, the equation of motion of the plate may be written as [3]:

$$\frac{D}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\}+2\frac{dD}{dr}\frac{d^3w}{dr^3}+\frac{1}{r}\left((2+\nu)\frac{dD}{dr}+r\frac{d^2D}{dr^2}\right)\frac{d^2w}{dr^2}-\frac{1}{r^2}\left(\frac{dD}{dr}-r\nu\frac{d^2D}{dr^2}\right)\frac{dw}{dr}$$
$$-\frac{k_s}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)+k_w w=-I\ddot{w} \tag{5}$$

where w is the transverse deflection and k_w and k_s are Winkler's and Pasternak's coefficients of the elastic foundation, respectively. ν is Poisson's ratio and ρ and D are the mass density and flexural rigidity of the plate, respectively:

$$D(r)=\frac{1}{1-\nu^2}\int_{-h/2}^{h/2}E(z,r)z^2dz \tag{6}$$

$$I(r)=\int_{-h/2}^{h/2}\rho(z,r)dz \tag{7}$$

Substituting Eq. (4) into Eqs. (6) and (7), leads to the following expressions:

$$D(r)=D^*A(m_1)e^{(n_1+3n_3)\frac{r}{b}} \tag{8}$$

$$I(r)=\int_{-h/2}^{h/2}\rho(r,z)dz=\rho_0 h_0 B(m_2)e^{(n_2+n_3)\frac{r}{b}} \tag{9}$$

where

$$A(m_1)=\frac{3}{m_1^3}\left[(4+(m_1-2)^2)e^{m_1}-4-(m_1+2)^2\right],\ B(m_2)=\left(\frac{e^{m_2}-1}{m_2}\right),\ D^*=\frac{E_0 h_0^{\ 3}}{12(1-\nu^2)} \tag{10}$$

Since Eq. (5) is linear with respect to the time differentiation operators, the time dependency of the solution may be expressed by an exponential function. Therefore, using a Kantorovich-type approximation, solution of Eq. (5) may be expressed in the following form:

$$w = f(r)\,e^{i\omega t} \tag{11}$$

where ω is the natural frequency and i =($\sqrt{-1}$) is the imaginary number. Substituting Eq. (11) into Eq. (5) yields:

$$\begin{aligned}&\frac{D}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)\right]\right\}+2D_{,r}\frac{d^3f}{dr^3}+\left(\frac{(2+\nu)D_{,r}+rD_{,rr}}{r}\right)\frac{d^2f}{dr^2}-\left(\frac{D_{,r}-r\nu D_{,rr}}{r^2}\right)\frac{df}{dr}\\&-\frac{k_s}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)+k_w f=\rho h\omega^2 f\end{aligned} \tag{12}$$

To present a more general solution, the following dimensionless parameters are defined:

$$\bar{r}=\frac{r}{b},\qquad \bar{f}=\frac{f}{h} \tag{13}$$

In the remaining part of the paper the bar symbol ($-$) will not be shown for the sake of simplicity. Therefore, Eq. (12) may be rewritten in terms of the dimensionless parameters defined in Eq. (13) as:

$$\begin{aligned}&\frac{D}{D^*}r^3\frac{d^4f}{dr^4}+\frac{r^2}{D^*}2(D+r\frac{dD}{dr})\frac{d^3f}{dr^3}+\frac{r}{D^*}\left(-D+(2+\nu)r\frac{dD}{dr}+r^2\frac{d^2D}{dr^2}\right)\frac{d^2f}{dr^2}-K_s r^3\frac{d^2f}{dr^2}\\&+\frac{1}{D^*}\left(D-r\frac{dD}{dr}+r^2\nu\frac{d^2D}{dr^2}\right)\frac{df}{dr}-K_s r^2\frac{df}{dr}+K_w r^3 f=r^3\Omega^2\left(\frac{\rho_c+\rho_m g}{\rho_c(g+1)}\right)f\end{aligned} \tag{14}$$

and Ω, K_w and K_s are the dimensionless natural frequency, dimensionless Winkler's normal stiffness of the foundation and dimensionless Pasternak's shear stiffness of the foundation, respectively:

$$\Omega^2=\frac{\rho_0 h_0 b^4}{D^*}\omega^2,\ \ K_s=\frac{k_s b^2}{D^*},\ \ K_w=\frac{k_w b^4}{D^*}. \tag{15}$$

By substituting the bending rigidity D from Eq.(8) into Eq. (14), Eq. (13) becomes:

$$\begin{aligned}&A(m_1)e^{(n_1+3n_3)r}r^3\frac{d^4f}{dr^4}+2[1+(n_1+3n_3)r]A(m_1)e^{(n_1+3n_3)r}r^2\frac{d^3f}{dr^3}+[(2+\nu)(n_1+3n_3)r\\&+(n_1+3n_3)^2r^2-1]A(m_1)e^{(n_1+3n_3)r}r\frac{d^2f}{dr^2}-K_s r^3\frac{d^2f}{dr^2}+[1-(n_1+3n_3)r+\nu(n_1+3n_3)^2r^2]\\&.A(m_1)e^{(n_1+3n_3)r}\frac{df}{dr}-K_s r^2\frac{df}{dr}+K_w r^3 f=B(m_2)e^{(n_2+n_3)r}r^3\Omega^2 f\end{aligned} \tag{16}$$

In the above equation it is assumed that f is the dimensionless deflection and r is the dimensionless radial coordinate.

Three type of the edge conditions are adopted in the present research: free, simply-supported, and clamped edge conditions. According to the classical plate theory, these edge conditions may be defined in terms of the dimensionless deflection function $f(r)$ at the edge of the circular plate ($r=1$) as follows:

Free edge:

$$M_r\big|_{r=1} = -D\left[\frac{d^2 f}{dr^2} + \nu\left(\frac{1}{r}\frac{df}{dr}\right)\right] = 0 \tag{17}$$

$$V_r\big|_{r=1} = D\left(\frac{d^3 f}{dr^3} + \frac{1}{r}\frac{d^2 f}{dr^2} - \frac{1}{r^2}\frac{df}{dr}\right) + D_{,r}\left(\frac{d^2 f}{dr^2} + \frac{\nu}{r}\frac{df}{dr}\right) = 0 \tag{18}$$

Simply-supported edge:

$$f(1) = 0, \quad M_r\big|_{r=1} = -D\left[\frac{d^2 f}{dr^2} + \nu\left(\frac{1}{r}\frac{df}{dr}\right)\right] = 0 \tag{19}$$

Clamped edge:

$$f(1) = 0, \quad \frac{df}{dr}\Big|_{r=1} = 0. \tag{20}$$

where M_r is the radial bending moment per unit length, and V_r is the effective radial shear force per unit length.

It is evident that since Eq. (16) is a fourth-order differential equation, then four boundary conditions are required to obtain the relevant integration constants. One may obtain two of those from the boundary conditions of the outer edge of the circular plate. However, the remaining two conditions may be determined from the regularity conditions at the center of the plate. In the case of an axisymmetric material distribution and axisymmetric boundary conditions, one may express the regularity conditions at the center ($r=0$) of the circular plate as follows:

$$\frac{df}{dr}\Big|_{r=0} = 0, \quad V_r\big|_{r=0} = \left\{D\left(\frac{d^3 f}{dr^3} + \frac{1}{r}\frac{d^2 f}{dr^2} - \frac{1}{r^2}\frac{df}{dr}\right) + D_{,r}\left(\frac{d^2 f}{dr^2} + \frac{\nu}{r}\frac{df}{dr}\right)\right\}\Big|_{r=0} = 0 \tag{21}$$

4 Employing the differential transform technique

4.1 A brief description of the differential transformation method

The differential transformation method (DTM), which was first proposed by Zhou [23] based on the Taylor's series expansion, is a semi-analytical method that has been proposed mainly for solving ordinary and partial differential equations.

In the DTM, certain transformation rules are applied and the governing differential equations and the relevant boundary conditions of the system are transformed into a set of algebraic equations in terms of the differential transforms of the original functions. Solution of these algebraic equations gives the desired solution of the problem. The basic definitions and the procedure of employing the method are introduced in the present section.

Consider a function $f(r)$ which is analytic in a domain R and let $r=r_0$ represent any point in R. It is intended to represent the function $f(r)$ by a power series whose center is located at $r=r_0$. The differential transform of the kth derivative of the function $f(r)$ is given by

$$F_k = \frac{1}{k!}\left[\frac{d^k f(r)}{dr^k}\right]_{r=r_0} \tag{22}$$

where F_k is the transformed function. The inverse transformation of the function *f(r)* is defined by

$$f(r) = \sum_{k=0}^{\infty}(r-r_0)^k F_k \tag{23}$$

Combining Eqs. (22) and (23) leads to:

$$f(r) = \sum_{k=0}^{\infty}\frac{(r-r_0)^k}{k!}\left[\frac{d^k f(r)}{dr^k}\right]_{r=r_0} \tag{24}$$

According to Eq. (24), it may be noted that the concept of the differential transform is derived from Taylor's series expansion. However, the method does not evaluate the derivatives symbolically. In this method, the lower case letter represents the original function and the upper case letter indicates the transformed function. In practical applications, the function *f(r)* is usually expressed by a finite series. Therefore, Eq. (24) may be rewritten as

$$f(r) = \sum_{k=0}^{N}\frac{(r-r_0)^k}{k!}\left[\frac{d^k f(r)}{dr^k}\right]_{r=r_0} \tag{25}$$

which implies that $\sum_{k=N+1}^{\infty}(r-r_0)^k F_k$ has to be negligibly small. In the present research, the N value is so chosen that the calculated natural frequencies converge.

In table 1, the basic mathematical expressions that are frequently subjected to the mentioned transformation in the practical problems, are listed along with their transformed expressions.

4.2 Transformation of the present governing equation

According to the basic transformation operations introduced in table 1, the transformed form of the governing Eq. (16) around $r_0=0$, may be obtained as:

$$\begin{aligned}
&A(m_1)\sum_{l=0}^{k}\sum_{i=0}^{k-l}\frac{(n_1+3n_3)^l}{l!}\delta(i-3)\frac{(k-l-i+4)!}{(k-l-i)!}F_{k-l-i+4} - K_s\sum_{i=0}^{k}\delta(i-3)\frac{(k-i+2)!}{(k-i)!}F_{k-i+2}\\
&+2A(m_1)\sum_{l=0}^{k}\sum_{i=0}^{k-l}\frac{(n_1+3n_3)^l}{l!}\big(\delta(i-2)+(n_1+3n_3)\delta(i-3)\big)\frac{(k-l-i+3)!}{(k-l-i)!}F_{k-l-i+3}\\
&+A(m_1)\sum_{l=0}^{k}\sum_{i=0}^{k-l}\frac{(n_1+3n_3)^l}{l!}\big((2+\nu)(n_1+3n_3)\delta(i-2)+(n_1+3n_3)^2\delta(i-3)-\delta(i-1)\big)\frac{(k-l-i+2)!}{(k-l-i)!}F_{k-l-i+2}\\
&+A(m_1)\sum_{l=0}^{k}\sum_{i=0}^{k-l}\frac{(n_1+3n_3)^l}{l!}\big(1-(n_1+3n_3)\delta(i-1)+\nu(n_1+3n_3)^2\delta(i-2)\big)\frac{(k-l-i+1)!}{(k-l-i)!}F_{k-l-i+1}\\
&-K_s\sum_{i=0}^{k}\delta(i-2)\frac{(k-i+1)!}{(k-i)!}F_{k-i+1}+K_w\sum_{i=0}^{k}\delta(i-3)F_{k-i} = B(m_2)\Omega^2\sum_{l=0}^{k}\sum_{i=0}^{k-l}\frac{(n_2+n_3)^l}{l!}\delta(i-3)F_{k-l-i}
\end{aligned} \tag{26}$$

Eq. (26) may be simplified to be:

$$
\begin{aligned}
& A(m_1)\sum_{l=0}^{k+4}\frac{(n_1+3n_3)^l}{l!}(k-l+2)^2(k-l+4)^2 F_{k-l+4} \\
& + A(m_1)\sum_{l=0}^{k+3}\frac{(n_1+3n_3)^{l+1}}{l!}\big[2(k-l+1)(k-l+2)+(k-l+2)(2+\nu)-1\big](k-l+3)F_{k-l+3} \\
& + A(m_1)\sum_{l=0}^{k+2}\frac{(n_1+3n_3)^{l+2}}{l!}\big[k-l+1+\nu\big](k-l+2)F_{k-l+2} - K_s(k-l+1)(k-l+2)F_{k-l+2} \\
& - K_s(k+2)F_{k+2} + K_w F_k = B(m_2)\sum_{l=0}^{k}\frac{(n_2+n_3)^l}{l!}\Omega^2 F_{k-l}
\end{aligned}
\tag{27}
$$

Therefore, the following equation can be obtained for *k=4, 5, 6, . . ., n* :

$$
\begin{aligned}
F_{k+4} = & \frac{1}{A(m_1)(k+2)^2(k+4)^2}\Big\{-A(m_1)\sum_{l=1}^{k+4}\frac{(n_1+3n_3)^l}{l!}(k-l+2)^2(k-l+4)^2 F_{k-l+4} \\
& - A(m_1)\sum_{l=0}^{k+3}\frac{(n_1+3n_3)^{l+1}}{l!}\Big[2(k-l)^2+8(k-l)+\nu(k-l+2)+7\Big](k-l+3)F_{k-l+3} \\
& - A(m_1)\sum_{l=0}^{k+2}\frac{(n_1+3n_3)^{l+2}}{l!}\big[k-l+1+\nu\big](k-l+2)F_{k-l+2} + B(m_2)\sum_{l=0}^{k}\frac{(n_2+n_3)^l}{l!}\Omega^2 F_{k-l} \\
& + K_s(k+2)^2 F_{k+2} - K_w F_k\Big\}
\end{aligned}
\tag{28}
$$

4.3 Transformation of the boundary/regularity conditions

Applying the transformation rules mentioned in table 1 to the boundary conditions of the edge (r = 1) of the circular plate, presented in Eqs. (17-20), the following equations are obtained:
Free edge condition:

$$\sum_{k=0}^{\infty}[k(k-1)+\nu k]F_k = 0,\quad \sum_{k=0}^{\infty}[k^2(k-2)+(n_1+3n_3)k(k-1+\nu)]F_k = 0 \tag{29}$$

Simply-supported edge condition:

$$\sum_{k=0}^{\infty}F_k = 0,\quad \sum_{k=0}^{\infty}[k(k-1)+\nu k]F_k = 0 \tag{30}$$

Clamped edge condition:

$$\sum_{k=0}^{\infty}F_k = 0,\quad \sum_{k=0}^{\infty}kF_k = 0 \tag{31}$$

At the center of the circular plate (r = 0), the regularity conditions (21) can be transformed as follows:

$$F_1 = 0, F_3 = -\frac{2(n_1+3n_3)}{9}(1+\nu)F_2 \tag{32}$$

5 Determination of the natural frequencies and the modal stresses

The natural frequencies may be extracted from the characteristics equations when the transformed boundary conditions (29-31) are incorporated. As it has been mentioned before, three kinds of edge conditions are considered in the present research: free, simply-supported, and clamped.

Obtaining the Taylor series expansion ($F_k,\ k \ge 4$) from Eq. (28) and substituting it into the boundary conditions (29-31), one may obtain the following expressions for each condition:

$$\begin{aligned} \Psi_{11}^{(n)}(\Omega)F_0 + \Psi_{12}^{(n)}(\Omega)F_2 = 0 \\ \Psi_{21}^{(n)}(\Omega)F_0 + \Psi_{22}^{(n)}(\Omega)F_2 = 0 \end{aligned} \tag{33}$$

where Ψ_{11}, Ψ_{12}, Ψ_{21} and Ψ_{22} are polynomials in terms of Ω corresponding to nth term. It may be readily seen that in Eq. (33), Ψ_{11}, Ψ_{12}, Ψ_{21} and Ψ_{22} represent series expressions. Eq. (33) can be expressed in the following matrix form:

$$\begin{bmatrix} \Psi_{11}^{(n)}(\Omega) & \Psi_{12}^{(n)}(\Omega) \\ \Psi_{21}^{(n)}(\Omega) & \Psi_{22}^{(n)}(\Omega) \end{bmatrix} \begin{Bmatrix} F_0 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{34}$$

Existence condition of the non-trivial solutions yields the following characteristic determinant:

$$\begin{vmatrix} \Psi_{11}^{(n)}(\Omega) & \Psi_{12}^{(n)}(\Omega) \\ \Psi_{21}^{(n)}(\Omega) & \Psi_{22}^{(n)}(\Omega) \end{vmatrix} = 0 \tag{35}$$

which may be used to calculate the dimensionless frequencies. To determine value of the jth natural frequency, the following convergence criterion may be used:

$$\frac{\left|\Omega_j^{(n)} - \Omega_j^{(n-1)}\right|}{\left|\Omega_j^{(n)}\right|} \le \varepsilon \quad , \; j = 1,2,3,...,n \tag{36}$$

where n is the iteration counter, Ω_j is the estimated value of the jth dimensionless natural frequency, and ε is a sufficiently small number that is chosen as $\varepsilon = 0.0001$ in the present study.

On the other hand, according to the classical plate theory, the radial and circumferential stresses may be related to the lateral deflection through the following equations:

$$\begin{aligned} \sigma_r &= \frac{E(r,z)z}{1-\nu^2}\left(w_{,rr} + \frac{\nu}{r} w_{,r} \right) \\ \sigma_\theta &= \frac{E(r,z)z}{1-\nu^2}\left(\frac{1}{r} w_{,r} + \nu w_{,rr} \right) \end{aligned} \tag{37}$$

Therefore, the dimensionless radial modal stresses may be calculated based on the following equation:

$$\begin{aligned} \bar{\sigma}_r &= \frac{b^2 \sigma_r}{h_0^2 E_0} = \frac{e^{m_1(z+\frac{1}{2})+n_1 r}}{1-\nu^2} z \left(w_{,rr} + \frac{\nu}{r} w_{,r} \right) \\ \bar{\sigma}_\theta &= \frac{b^2 \sigma_\theta}{h_0^2 E_0} = \frac{e^{m_1(z+\frac{1}{2})+n_1 r}}{1-\nu^2} z \left(\frac{1}{r} w_{,r} + \nu w_{,rr} \right) \end{aligned} \tag{38}$$

6 Results and discussions

Example 1: As a validation example, the first three natural frequencies of homogeneous plates with uniform thickness and simply-supported, clamped, and free edge conditions already studied by Refs. [2] and [24] are reexamined. Leissa [24] used the series solution and Wu et al. [2] employed the generalized differential quadrature method to extract the results. Poisson’s ratio is assumed to be 0.3. Table 2, compares present results with results of Leissa

[24] and Wu et al. [2] for the dimensionless natural frequency $\Omega = \omega b^2 \sqrt{\frac{\rho_0 h_0}{D*}}$. Based on the results given in table 2, there is a good agreement among the results. Present results are almost coincident with results obtained by Wu et al. [2]. Therefore, accuracy of the present results is proven.

Example 2: To evaluate influence of each of the geometric and the through-the-thickness and in-plane material parameters on the natural frequencies of the plate for various edge conditions (simply-supported, clamped, and free edge conditions) clearly, effects of each parameter are studied individually. To this end, the plates considered in the foregoing example are considered again and their material properties or geometric specifications are changed accordingly. Results are given in tables 3 to 7 for the first two natural frequencies. As it has been mentioned earlier, the parameter n_3 describes variations of the thickness of the plate in the radial direction. Since the natural frequencies are dimensionless, the results may be used for any geometric aspect ratio. Initial values of the first two natural frequencies have been given in table 2. As it may be expected, while positive m_1 and n_1 parameters increase the natural frequencies, positive values of the m_2 and n_2 parameters decrease the natural frequencies. Therefore, if for example it is intended to change natural frequencies of the plate to avoid coincidence with natural frequencies of the adjacent components or the excitation frequency of the foundation, one may use a plate whose material properties are graded either in the transverse or radial direction. According to Eqs. (8) and (9), in contrast to the thickness variation parameter n_3 which appears in both equations, n_1 only appeared in Eq. (8) and thus positive n_1 values lead to stiffer plate and subsequently, higher frequencies. Furthermore, the mass density parameter n_2 is appeared in Eq. (9) only, and therefore its positive values decrease the natural frequencies. While according to Eq. (9), effect of identical n_2 and n_3 values on the mass moment of inertia is equivalent, effect of the n_3 parameter on the bending rigidity is three time that of n_1. Therefore, controlling the natural frequencies with parameters that have distinct effects (n_1 and n_3) is more desirable than using a single n_3 parameter that affects both the bending rigidity and mass moment of inertia of the plate.

Example 3: A sensitivity analysis is performed in the present example. In this regard, effects of seven geometric, material, and foundation parameters (m_1, m_2, n_1, n_2, n_3, K_w, and K_s) on the first two normalized natural frequencies of the two-directional functionally graded plate are investigated. Since various boundary conditions have been used in the present study, effects of the transverse variations of the material properties (m-parameters) are evaluated for plates with simply-supported edge conditions. Effects of the n-parameters are evaluated for clamped plates, and effects of the elastic foundation parameters (K-parameters) are studied for plates with free edges. Results are illustrated in Figs. 1 to 6. In each figure, variation of the normalized natural frequency versus one of the aforementioned parameters is plotted, for specific combinations of the other parameters. Moreover, in each figure, the baseline curve corresponds to (m_1=1, m_2=1, n_1=1, n_2=1, n_3=0.5, K_w=5, and K_s=25) excluding the parameter that is used for the relevant sensitivity analysis. In Figs. 1 to 5, in some cases, curves associated with different parameters are coincident. On the other hand, some curves that may seem to be completely or partially coincident with other curves in one vibration mode are not coincident in the other vibration mode.

Results depicted in Figs. 1 to 5 reveal that negative n_2 and m_2 values increase the natural frequencies. This conclusion confirms the previous conclusions. Effect of the shear stiffness of the elastic foundation (Pasternak's coefficient) on the fundamental natural frequency of the plate is noticeable whereas it has somewhat ignorable effect on the second natural frequency.

Influence of the shear stiffness of the elastic substrate is more remarkable for smaller m_1 and m_2 parameters.

To investigate effect of the stiffness of the elastic foundation on the natural frequencies of the plate more clearly, a plate with free edges is considered in the present sensitivity analysis. Results shown in Fig. 6 reveal that while effects of the Winkler and Pasternak parameters of the foundation (K_w and K_s) on the fundamental natural frequency of the plate are remarkable, effects of these parameters on the higher (second) vibration natural frequencies are smaller. Effect of K_w on the second natural frequency is almost ignorable.

Example 4: For a circular plate subjected to smooth variations in the applied transverse pressure, the maximum radial stress that is mainly larger than the maximum circumferential stress usually occurs in the center or edge of the plate. To enable a proper conclusion, the transverse distributions of the radial stress are plotted in Fig. 7 for sections correspond to ($r=0$, and $r=1$) of a plate with uniform thickness and clamped edges. Therefore, influence of the moment exerted by the boundary as well as effects of the elastic foundation are engaged. As is may be noticed from results appeared in Fig. 7, the maximum stresses of the mentioned sections have relatively different signs. However, in a modal analysis, the absolute sign of the mode shape itself is of no importance (the reversed mode shape denotes the same mode shape). Results show that a proper choosing of the material properties may lead to smaller stresses or somewhat more uniform stress distribution. For example, the case with ($m_1=-1$, $n_1=-1$) has led to smaller stress gradient in both critical sections. Therefore, by a proper adoption of the distribution of the material properties, a somewhat more uniform stress distribution or strengthening of the sections of the plate may be attained without the need to change the geometric parameters of the plate (taking into account the resulted allowable stresses). Even though it may seem that traditional materials with higher Young's modulus have generally higher mass densities, this is not a rule, e.g. in the composite materials, Young's modulus to mass density ratio is remarkably higher.

7 Conclusions

In the present paper, a semi-analytical solution is developed for free vibration and modal stress analyses of circular plates made of two-directional functionally graded materials resting on two-parameter elastic foundations. The differential transformation method is employed to develop the solution. The thickness of the plate may vary in the radial direction. The solution covers complex combinations of the material properties, edge conditions, and parameters of the elastic foundation. Therefore, the resulted non-dimensional solution may be used for a wide range of the practical problems. Influence of monitoring variations of the material properties on the natural frequencies and the through-the-thickness distribution of the modal stress is also investigated. Results reveal that by a proper adoption of the distribution of the material properties, a somewhat uniform stress distribution or strengthening of the sections of the plate may be attained without the need to change the geometric parameters of the plate. Many novelties are included in the present research. Comparisons made between the present results and results reported by well-known references for special cases treated before, have confirmed accuracy and efficiency of the present approach.

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Nomenclature

A, B	compact expressions
b	outer radius of the plate
D, D^*	flexural rigidity, reference flexural rigidity
E, E_0	Young's modulus, reference Young's modulus
F_k	transformed function
f	amplitude of the lateral deflection
h, h_0	thickness, thickness at the center of the plate
I	mass moment of inertia
i	the imaginary number and a counter
K_w, K_S	non-dimensional Winkler's and Pasternak's coefficients
k	counter
k_w, k_s	Winkler's and Pasternak's coefficients of the elastic foundation
l	counter
M_r	radial bending moment per unit length
m_1, m_2, m_3	Young's, density, and thickness exponents of the power law functions
N	upper limit of the series counter
n	iteration counter
n_1, n_2, n_3	Young's, density, and thickness exponents of the power law functions
r	radius
t	time
V_r	effective shear force per unit length
w	lateral deflection
z	transverse coordinates

Greek symbols

δ	Kronecker delta function
ε	convergence tolerance
ν	Poisson's ratio
ρ, ρ_0	mass density, reference mass density
σ_r,σ_θ	radial, circumferential stresses
Ω	dimensionless natural frequency
ω	natural frequency
Ψ	polynomials in terms of the natural frequencies

Table 1 Some of the transformation rules of the one-dimensional DTM.

Original function	Transformed function
$f(r)=g(r)\pm h(r)$	$F_k = G_k \pm H_k$
$f(r)=\lambda g(r)$	$F_k = \lambda G_k$
$f(r)=g(r)\cdot h(r)$	$F_k = \sum_{l=0}^{k} G_l \cdot H_{k-l}$
$f(r)=\dfrac{d^n g(r)}{dr^n}$	$F_k = \dfrac{(k+n)!}{k!} G_{k+n}$
$f(r)=r^n$	$F_k = \delta(k-n) = \begin{cases} 1 & k=n \\ 0 & k \neq n \end{cases}$
$f(r)=\exp(\lambda r)$	$\dfrac{\lambda^k}{k!}$

Table 2 The first three natural frequencies of isotropic plates with uniform thickness ($m_1, m_2, n_3, K_s, K_w = 0$).

			$n_1=-0.5$			$n_1=0$			$n_1=1$		
			$n_2=-0.5$	$n_2=0$	$n_2=1$	$n_2=-0.5$	$n_2=0$	$n_2=1$	$n_2=-0.5$	$n_2=0$	$n_2=1$
Simply support	Ω_1	present	4.9674	4.4630	3.5491	5.4854	4.9351	3.9330	7.0874	6.3903	5.1187
		Ref[3]	4.8363	4.3455	3.4540	5.4854	4.9351	3.9330	7.0874	6.3917	5.1187
	Ω_2	present	29.5661	26.1629	20.3791	33.4691	29.7200	23.2937	43.0951	38.5045	30.5755
		Ref[3]	29.4088	26.0326	20.2762	33.4691	29.7200	23.2937	43.0951	38.5132	30.5755
	Ω_3	present	73.9182	65.2549	50.5484	83.7329	74.1561	57.7665	107.172	95.4506	75.2496
		Ref[3]	73.7511	65.1256	50.4382	83.7329	74.1561	57.7665	107.1722	95.4730	75.2496
clamp	Ω_1	present	9.56842	8.7401	7.2250	11.1468	10.2163	8.4748	15.3791	14.1597	11.8597
		Ref[3]	9.5005	8.6879	7.1733	11.1464	10.2158	8.4746	15.3791	14.1597	11.8597
	Ω_2	present	39.0448	34.7307	27.2680	44.5765	39.7732	31.4124	58.2294	52.2669	41.7754
		Ref[3]	38.9153	34.6170	27.1791	44.5753	39.7711	31.4115	58.2294	52.2669	41.7754
	Ω_3	present	88.3185	78.1659	60.7654	100.389	89.1086	69.6632	129.3342	115.4552	91.2929
		Ref[3]	88.1704	78.0372	60.6645	100.3867	89.1041	69.6613	129.3342	115.4552	91.2929
free	Ω_1	present	9.36847	8.10603	6.05646	10.3964	9.0031	6.7390	13.1162	11.4019	8.5971
		Ref[3]	9.2861	8.0258	5.9841	10.3962	9.0031	6.7388	13.1162	11.4019	8.5971
	Ω_2	present	38.7155	34.0272	26.1711	43.6095	38.4432	29.7518	55.3258	49.0768	38.4637
		Ref[3]	38.5853	33.9100	26.0797	43.6084	38.4432	29.7511	55.3258	49.0768	38.4637
	Ω_3	present	87.9651	77.4500	59.6732	99.3810	87.7502	68.0057	126.2877	112.1630	87.9488
		Ref[3]	87.8164	77.3162	59.5707	99.3786	87.7502	68.0039	126.2877	112.1630	87.9488

Table 3 Influence of the parameter m_1 that describes transverse variations of Young's modulus, on the first two natural frequencies of the plate ($m_2=n_1=n_2=n_3=K_s=K_w=0$).

m_1	Simply support		clamp		free	
	Ω_1	Ω_2	Ω_1	Ω_2	Ω_1	Ω_2
-2	3.436	20.697	7.114	27.697	6.269	26.772
-1.75	3.548	21.370	7.345	28.598	6.473	27.643
-1.5	3.675	22.136	7.609	29.623	6.705	28.634
-1.25	3.821	23.010	7.909	30.792	6.970	29.764
-1	3.987	24.010	8.253	32.131	7.273	31.058
-0.75	4.177	25.157	8.647	33.665	7.620	32.541
-0.5	4.396	26.473	9.099	35.426	8.019	34.243
-0.25	4.647	27.984	9.619	37.449	8.477	36.198
-0.1	4.815	28.997	9.967	38.803	8.784	37.508
0.1	5.061	30.483	10.478	40.793	9.234	39.431
0.25	5.265	31.710	10.900	42.435	9.606	41.018
0.5	5.644	33.992	11.684	45.488	10.297	43.969
0.75	6.078	36.603	12.582	48.983	11.088	47.347
1	6.573	39.587	13.607	52.975	11.992	51.206
1.25	7.138	42.989	14.777	57.528	13.022	55.607
1.5	7.781	46.863	16.108	62.712	14.196	60.618
1.75	8.512	51.265	17.621	68.603	15.530	66.312
2	9.342	56.261	19.338	75.288	17.043	72.774

Table 4 Influence of the parameter m_2 that describes transverse variations of the mass density, on the first two natural frequencies of the plate ($m_1=n_1=n_2=n_3=K_s=K_w=0$).

m_2	Simply support		clamp		free	
	Ω_1	Ω_2	Ω_1	Ω_2	Ω_1	Ω_2
-2	7.505	45.200	15.536	60.486	13.692	58.467
-1.75	7.182	43.253	14.867	57.881	13.102	55.948
-1.5	6.857	41.297	14.195	55.263	12.510	53.418
-1.25	6.532	39.337	13.521	52.641	11.916	50.883
-1	6.207	37.380	12.849	50.022	11.323	48.352
-0.75	5.883	35.433	12.179	47.416	10.733	45.833
-0.5	5.563	33.502	11.516	44.833	10.149	43.336
-0.25	5.246	31.595	10.860	42.281	9.571	40.869
-0.1	5.059	30.466	10.472	40.769	9.229	39.408
0.1	4.812	28.980	9.961	38.781	8.779	37.486
0.25	4.630	27.883	9.584	37.312	8.446	36.067
0.5	4.332	26.091	8.968	34.916	7.904	33.750
0.75	4.043	24.353	8.371	32.589	7.377	31.500
1	3.764	22.672	7.793	30.340	6.868	29.327
1.25	3.496	21.055	7.237	28.176	6.378	27.236
1.5	3.239	19.507	6.705	26.104	5.909	25.233
1.75	2.994	18.030	6.197	24.128	5.462	23.322
2	2.761	16.628	5.715	22.251	5.037	21.508

Table 5 Influence of the parameter n_1 that describes radial variations of the Young's modulus, on the first two natural frequencies of the plate ($m_1 = m_2 = n_2 = n_3 = K_s = K_w = 0$).

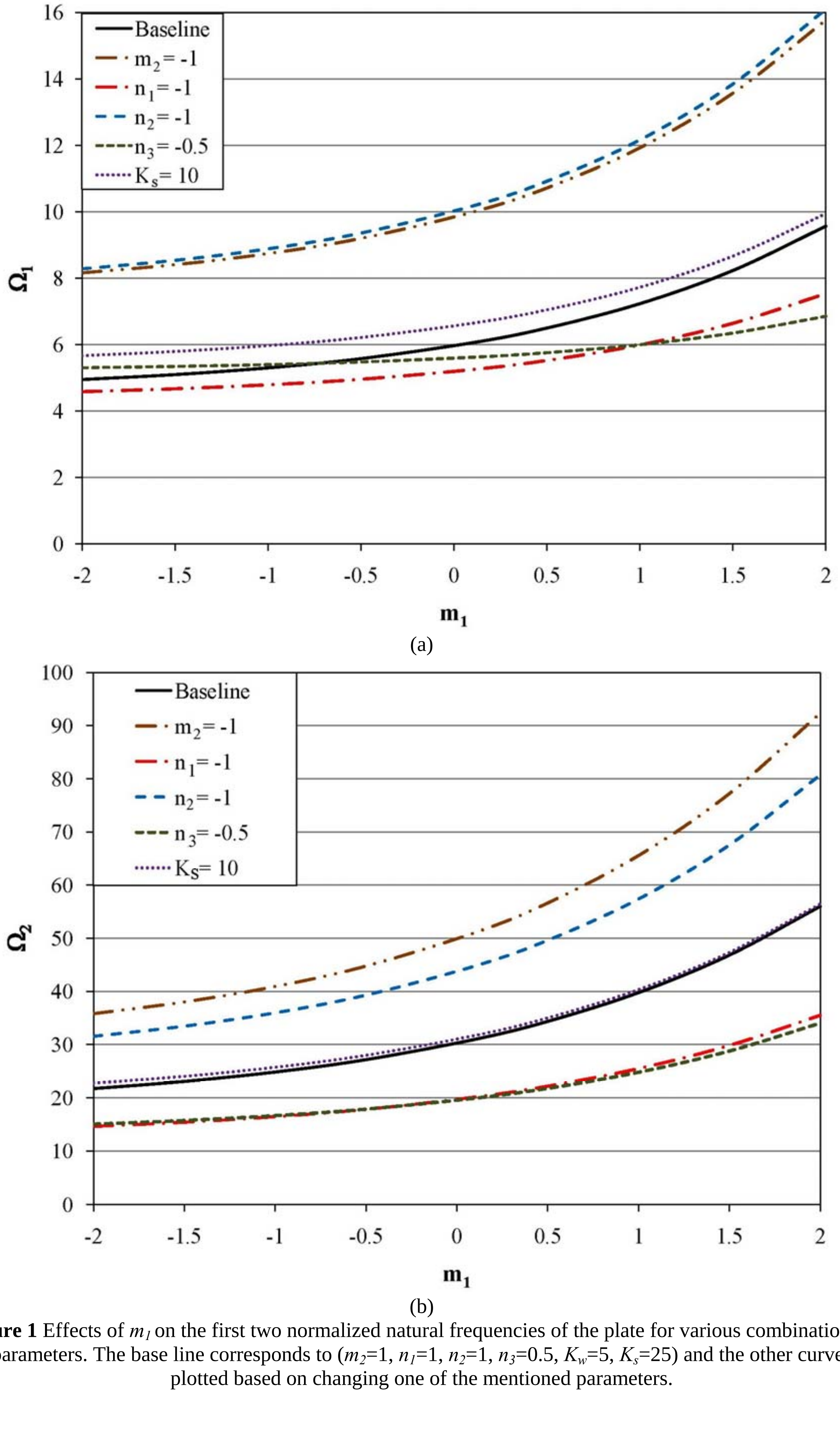
n_1	Simply support		clamp		free	
	Ω_1	Ω_2	Ω_1	Ω_2	Ω_1	Ω_2
-1	4.38640	23.4060	7.74010	30.6430	7.55643	30.4005
-0.75	4.37316	24.6857	8.18267	32.5661	7.78380	32.1107
-0.25	4.65628	27.8517	9.43227	37.1361	8.51600	36.1410
-0.1	4.81455	28.9515	9.89106	38.6902	8.79993	37.5008
0.1	5.06595	30.5151	10.5548	40.8868	9.21634	39.4128
0.25	5.27744	31.7537	11.0883	42.6229	9.55260	40.9151
0.5	5.65504	33.9240	12.0396	45.6727	10.1477	43.5329
0.75	6.03732	36.1932	13.0635	48.8968	10.7705	46.2661
1.25	6.68258	40.8237	15.3354	55.7498	12.0220	51.9187
1.5	6.86945	43.0492	16.6087	59.3075	12.6088	54.7352

Table 6 Influence of the parameter n_2 that describes radial variations of the mass density, on the first two natural frequencies of the plate ($m_1 = m_2 = n_1 = n_3 = K_s = K_w = 0$).

n_2	Simply support		clamp		free	
	Ω_1	Ω_2	Ω_1	Ω_2	Ω_1	Ω_2
-1.5	6.675	42.184	13.115	55.551	13.858	55.889
-1.25	6.367	39.844	12.610	52.629	12.896	52.553
-1	6.066	37.614	12.113	49.827	12.002	49.401
-0.75	5.772	35.490	11.625	47.144	11.170	46.422
-0.25	5.206	31.546	10.676	42.118	9.675	40.951
-0.1	5.042	30.439	10.398	40.697	9.266	39.429
0.1	4.828	29.015	10.034	38.862	8.747	37.479
0.25	4.672	27.984	9.765	37.529	8.376	36.076
0.5	4.417	26.337	9.324	35.391	7.792	33.843
0.75	4.170	24.775	8.894	33.353	7.247	31.736
1.25	3.703	21.889	8.065	29.563	6.263	27.879
1.5	3.483	20.559	7.668	27.806	5.819	26.115

Table 7 Influence of the parameter n_3 that describes variations of the thickness of the plate in the radial direction, on the first two natural frequencies of the plate ($m_1 = m_2 = n_1 = n_2 = K_s = K_w = 0$).

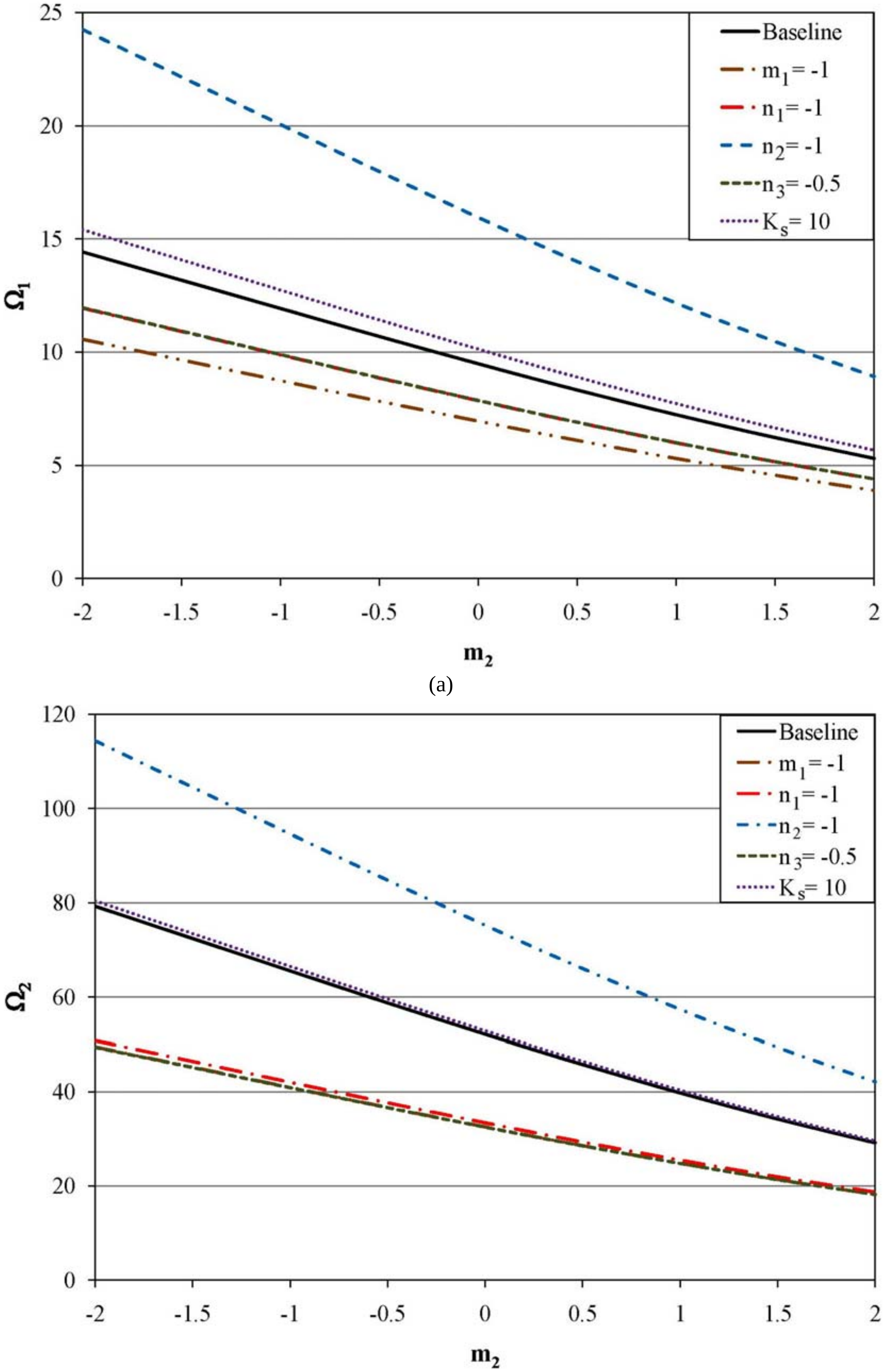
n_3	Simply support		clamp		free	
	Ω_1	Ω_2	Ω_1	Ω_2	Ω_1	Ω_2
-0.5	5.218	24.3834	7.955	31.110	8.457	31.552
-0.4	4.872	24.934	8.057	32.267	8.326	32.456
-0.3	4.663	25.756	8.350	33.726	8.321	33.613
-0.2	4.611	26.843	8.824	35.478	8.440	35.011
-0.1	4.711	28.176	9.456	37.503	8.675	36.631
0.1	5.236	31.426	11.075	42.245	9.396	40.405
0.2	5.564	33.237	12.014	44.881	9.822	42.468
0.3	5.864	35.083	13.021	47.630	10.247	44.574
0.4	6.083	36.883	14.098	50.441	10.635	46.658
0.5	6.164	38.542	15.265	53.255	10.949	48.642



(a)

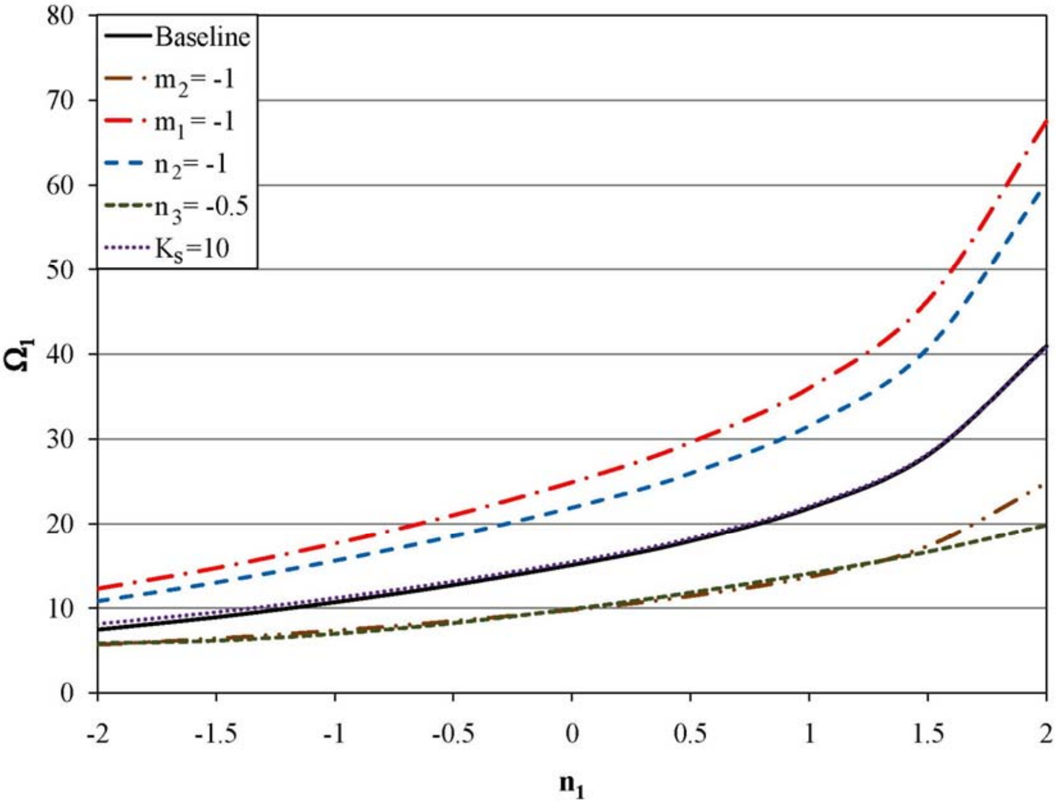
(b)

Figure 1 Effects of m_1 on the first two normalized natural frequencies of the plate for various combinations of the parameters. The base line corresponds to (m_2=1, n_1=1, n_2=1, n_3=0.5, K_w=5, K_s=25) and the other curves are plotted based on changing one of the mentioned parameters.

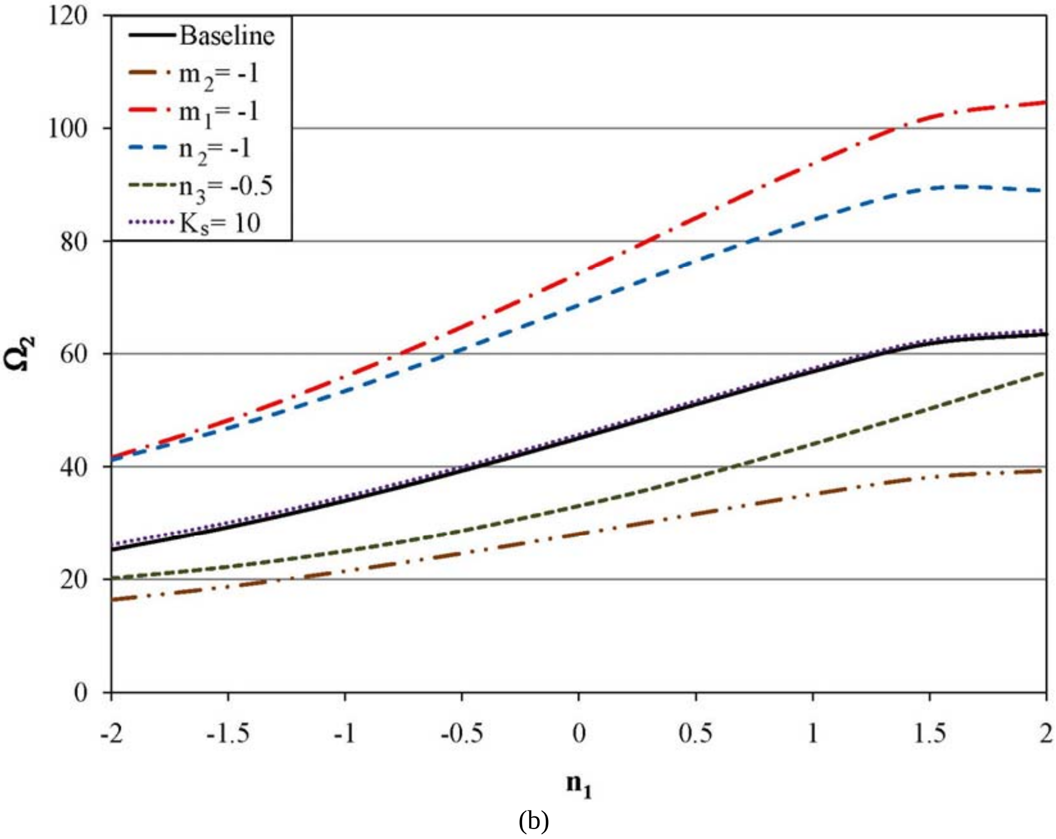


(b)

Figure 2 Effects of m_2 on the first two normalized natural frequencies of the plate for various combinations of the parameters. The base line corresponds to (m_1=1, n_1=1, n_2=1, n_3=0.5, K_w=5, K_s=25) and the other curves are plotted based on changing one of the baseline parameters.

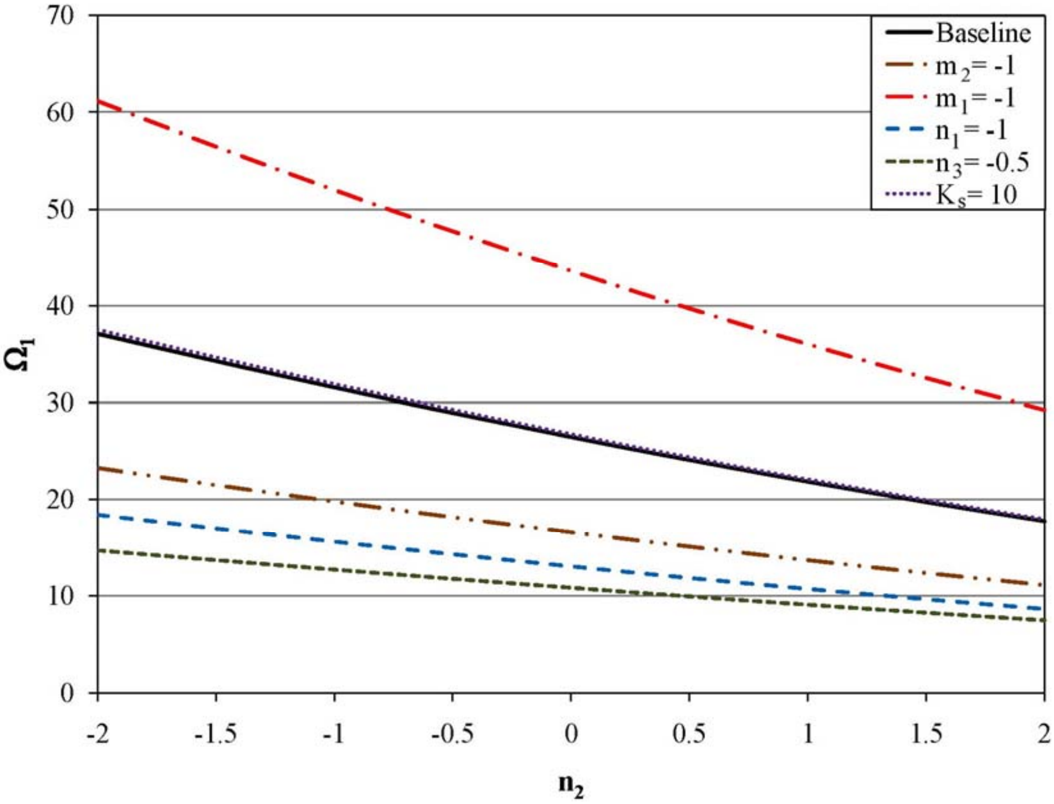


(a)

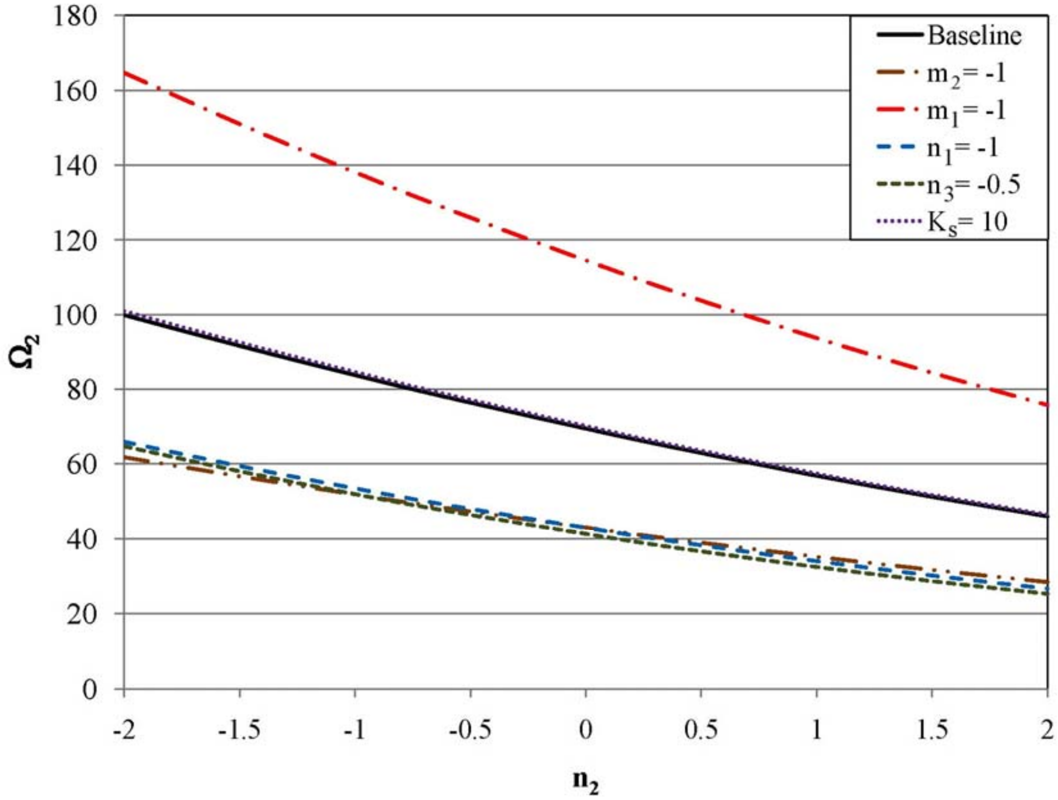


(b)

Figure 3 Influence of n_1 on the first two normalized natural frequencies of the plate for various combinations of the parameters. The base line corresponds to (m_1=1, m_2=1, n_2=1, n_3=0.5, K_w=5, K_s=25) and the other curves are plotted based on changing one of the baseline parameters.

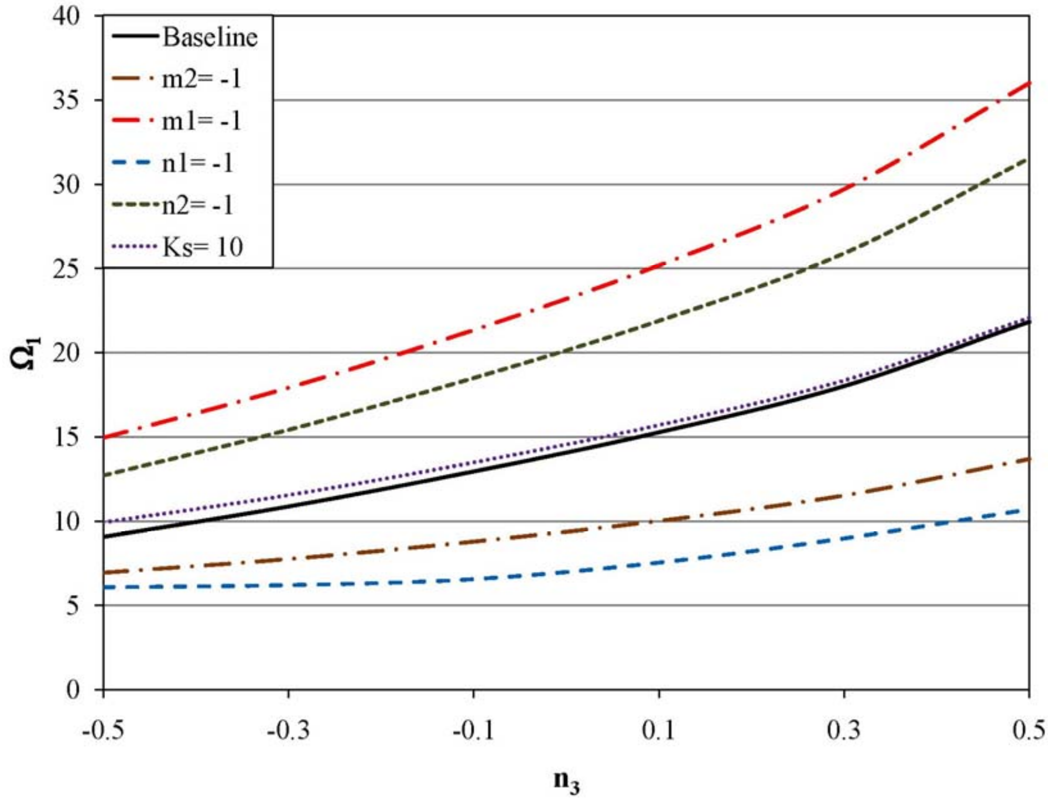


(a)

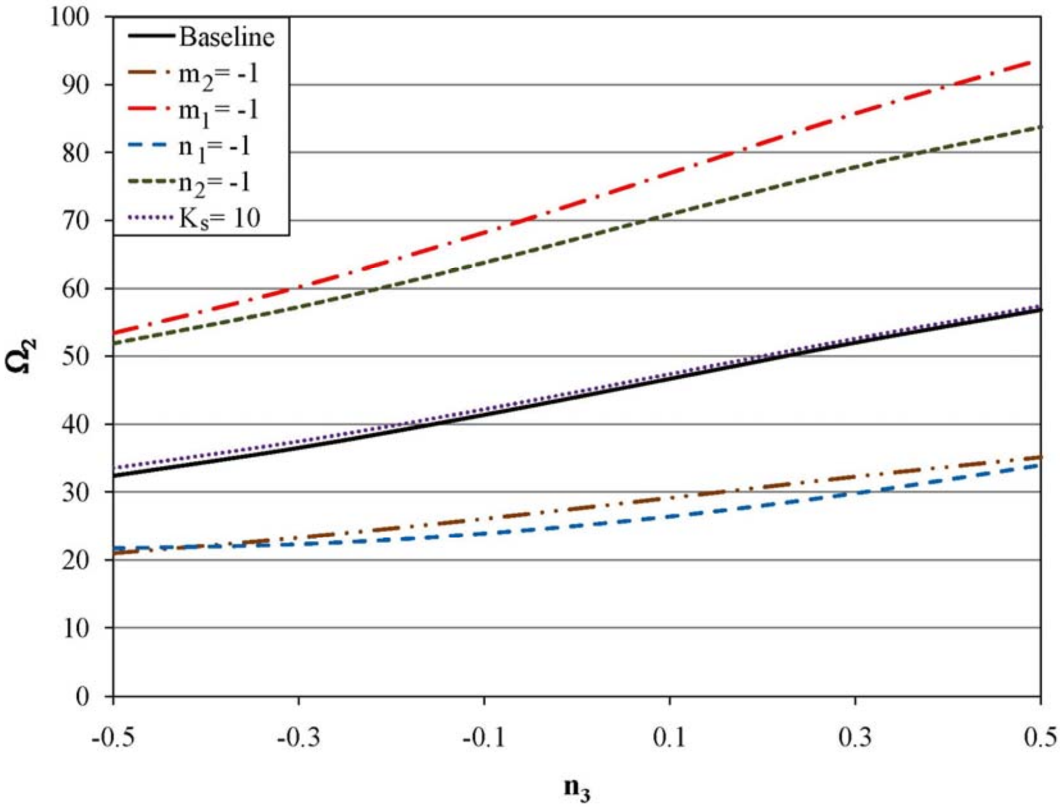


(b)

Figure 4 Influence of n_2 on the first two normalized natural frequencies of the plate for various combinations of the parameters. The base line corresponds to (m_1=1, m_2=1, n_1=1, n_3=0.5, K_w=5, K_s=25) and the other curves are plotted based on changing one of the baseline parameters.

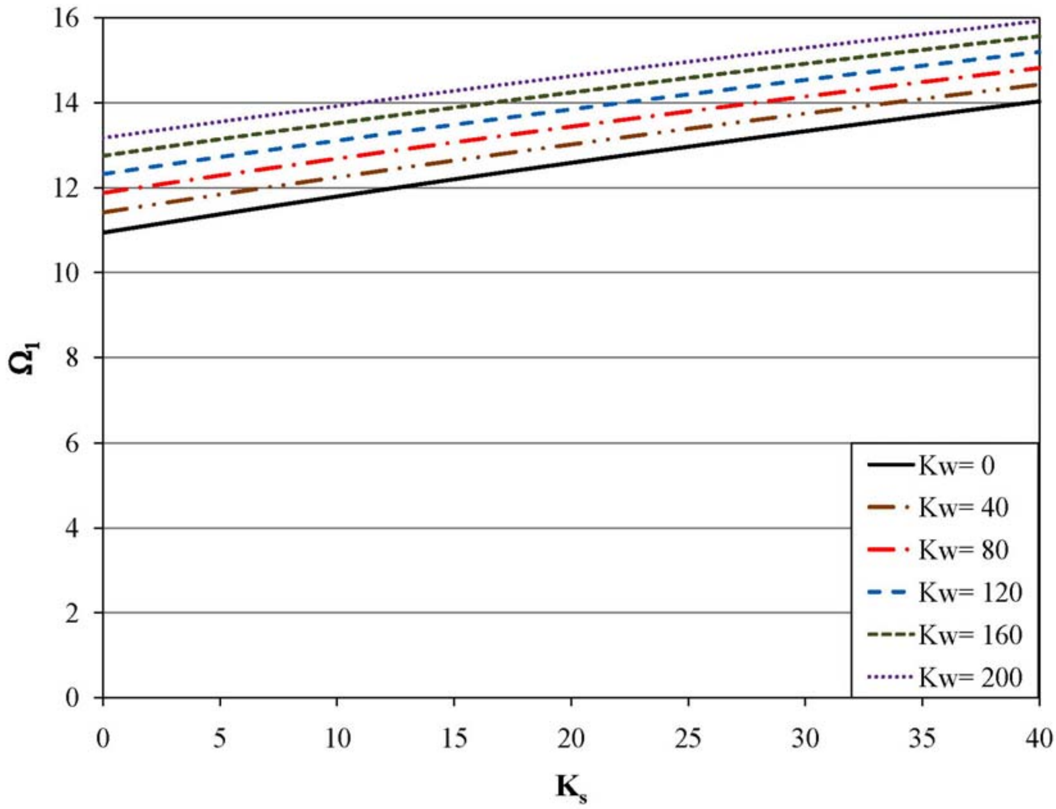


(a)

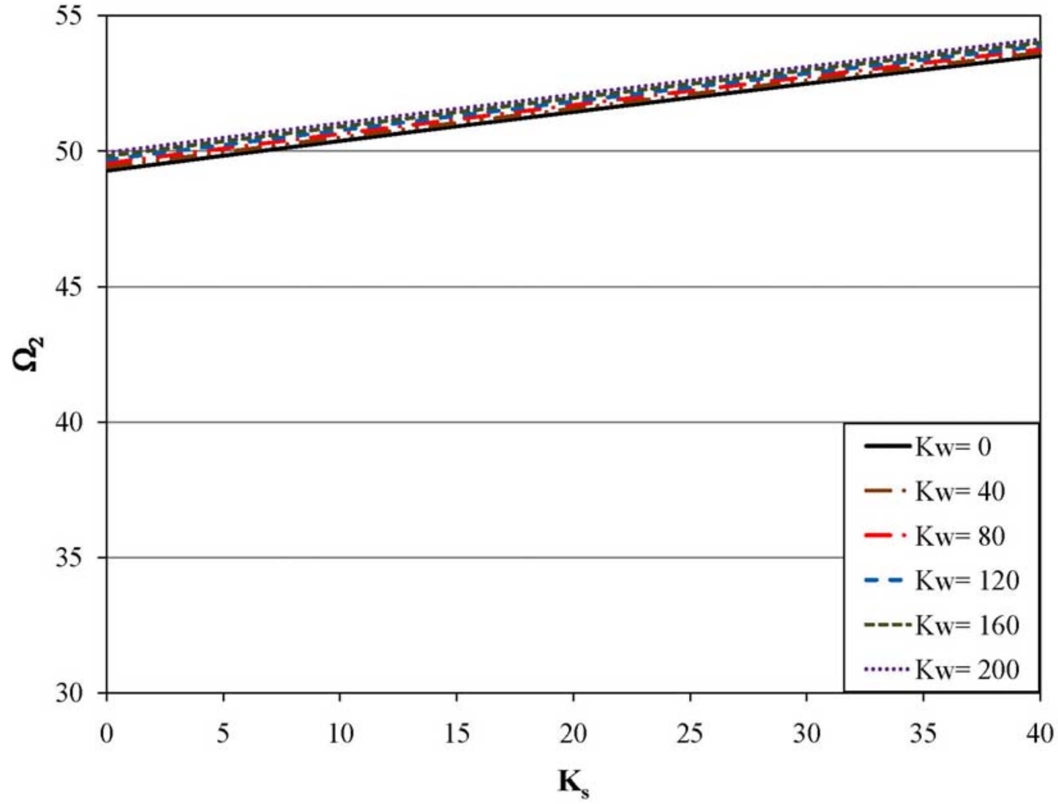


(b)

Figure 5 Effects of n_3 on the first two normalized natural frequencies of the plate for various combinations of the parameters. The base line corresponds to (m_1=1, m_2=1, n_1=1, n_2=1, K_w=5, K_s=25) and the other curves are plotted based on changing one of the baseline parameters.

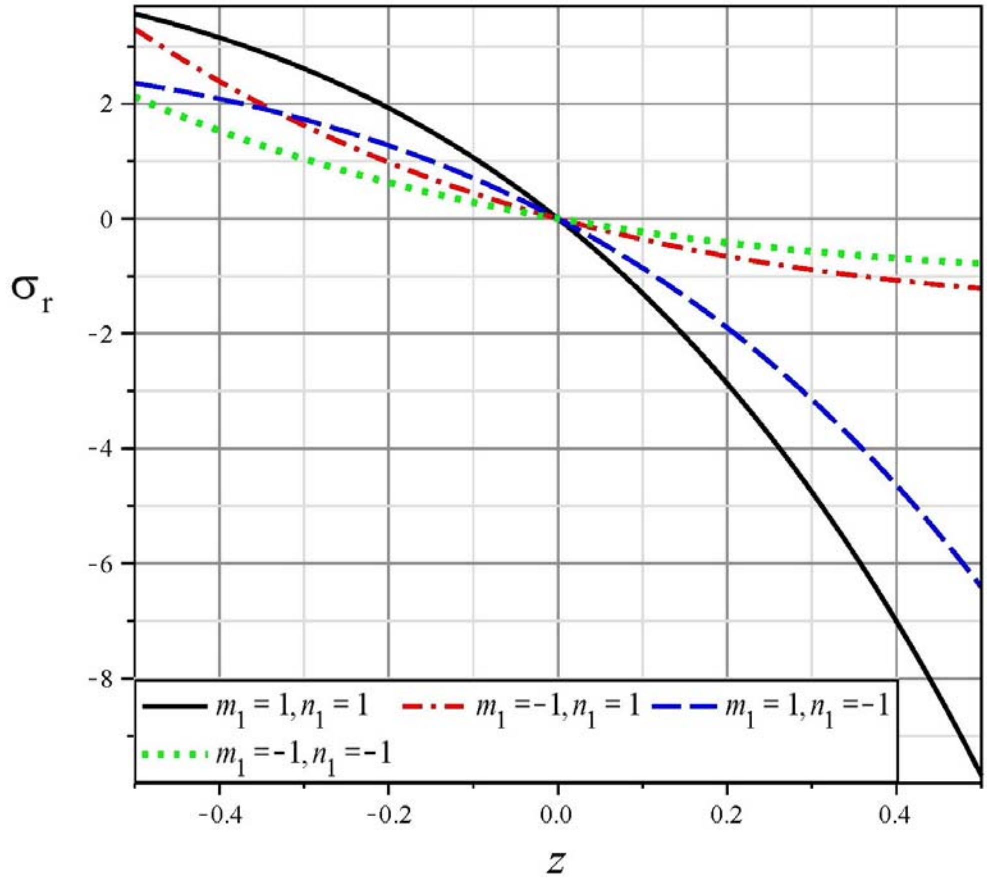


(a)

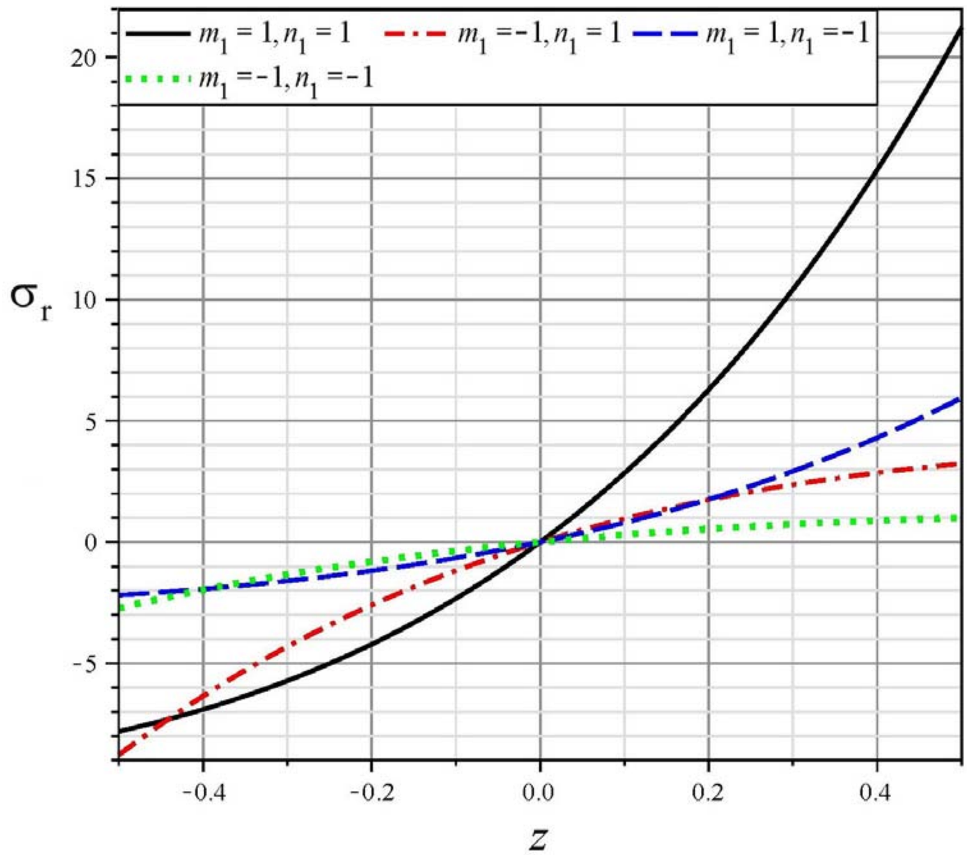


(b)

Figure 6 Effects of the parameters of the elastic foundation on the first two normalized natural frequencies of the plate with free edges for various combinations of the parameters (m_1=1, m_2=1, n_1=1, n_2=1, n_3=0.5).



(a)



(b)

Figure 7 Through-the-thickness distribution of the modal radial stress at: (a) *r=0* and (b) *r=1*.

چکیده

در مقاله کنونی، بر پایه روش تبدیل دیفرانسیلی، حلی نیمه تحلیلی برای تحلیل ارتعاش آزاد و تنشهای مودال ورقهای گرد ضخامت متغییر ساخته شده از مواد هدفمند دو جهته با لبه‌های مقید و تکیه‌گاه الاستیک ارائه شده است. تغییرات پارامترهای هندسی و مواد، توسط پنج تابع نمایی متفاوت کنترل شده‌اند. حل ارائه شده بر حسب پارامترهای بی بعد، می‌تواند ترکیبهای پیچیده‌ای از ویژگیهای مواد، شرایط لبه‌ای و ضرائب تکیه‌گاه الاستیک را پوشش دهد. نتایج آشکار می‌سازند که با انتخاب مناسب توزیع ویژگیهای مواد، می‌توان استحکام یکنواختی را برای مقاطع تامین نمود؛ بدون آنکه نیازی به تغییر پارامترهای هندسی ورق باشد.