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Modeling Diffusion to Thermal Wave Heat Propagation by Using Fractional Heat Conduction Constitutive Model

H. R. Ghazizadeh[*]
Graduate Student

M. Maerefat[†]
Associate Professor

Based on the recently introduced fractional Taylor's formula, a fractional heat conduction constitutive equation is formulated by expanding the single-phase lag model using the fractional Taylor's formula. Combining with the energy balance equation, the derived fractional heat conduction equation has been shown to be capable of modeling diffusion-to-Thermal wave behavior of heat propagation by changing the order of fractional differentiation. Finally, finite sine-Fourier and Laplace transforms are employed to find the exact solution of a signaling problem.

Keywords: Fractional calculus; Non-Fourier heat conduction; Generalized Taylor's Formula; Diffusion-to-Thermal wave propagation; Exact solution

1. Introduction

The relation between the heat flux vector and temperature gradient is called heat conduction constitutive model. The most well known constitutive relation in heat transfer is Fourier model which is originally based on experimental observations. This model which is pure diffusive in nature considers the instantaneous flow of heat in the medium in the presence of even a small temperature gradient. In other words, the velocity of Thermal propagation is infinite according to Fourier model which is in conflict with physical laws. Although it works well in many physical and engineering applications, many experimental studies have shown the inadequacy of Fourier model in some situations of practical interest. Up to now, some non-Fourier constitutive models have been introduced among which Cattaneo and phase-lagging models have found greater applications.

On the other hand, in the past three decades fractional calculus has proved its efficiency in modeling the intermediate anomalous behaviors observed in different physical phenomena. Fractional calculus is the calculus of differentiation and integration of non-integer orders. It is as old as classical (integer-order) calculus. However, until the recent decades it had not found considerable applications in practical and engineering fields and had been studied only in pure mathematics. Today, fractional calculus has shown great promise in different fields of science and engineering because of its inherent great abilities in modeling anomalous behaviors observed in many complex processes. Some main areas of application of fractional calculus

[*] Graduate Student, Mechanical Engineering Department, Tarbiat Modarres University, Tehran.
[†] Corresponding author, Associate Professor, Mechanical Engineering Department, Tarbiat Modarres University, Tehran, maerefat@modares.ac.ir

are modeling anomalous diffusion, viscoelasticity, finance, control theory and bioengineering [1-3].

In this study, by expanding the single-phase lag model with the recently introduced fractional Taylor series formula, a fractional Cattaneo constitutive model is obtained. Then by combining the fractional Cattaneo model with energy equation, a generalized heat equation is derived. Then, exact solutions for two examples of the proposed fractional heat conduction constitutive equation are presented.

Rest of the paper is organized as follows: In section 2, classical Cattaneo and also phase-lagging models are briefly introduced. In section 3, the concept of fractional heat conduction constitutive model is introduced and a generalized heat conduction equation is derived. In section 3, two illustrative problems including step and pulse heating are investigated and for the former case, exact solutions are obtained by using finite sine-Fourier and Laplace transforms and in terms of two-parameter Mittag-Leffler function and graphical results are presented for both cases. and finally, conclusion ends the note in section 4.

2. Integer-order non-Fourier constitutive models

In this section, Cattaneo and phase lagging non-Fourier models are briefly introduced. In Cattaneo model, in order to consider finite velocity of thermal perturbations, heat flux is relaxed through a relaxation time τ in the following form:

$$q(\vec{r},t) + \tau \frac{\partial q(\vec{r},t)}{\partial t} = -k\nabla T(\vec{r},t) \tag{1}$$

where $\tau = D/C_t^2$ is thermal relaxation time, k, D and C_t are thermal conductivity, thermal diffusivity and speed of thermal wave in the medium, respectively.

If we combine Eq. (1) with energy equation

$$\rho c \frac{\partial T}{\partial t} = -\mathrm{div}\left(q(\vec{r},t)\right) + q_{gen} \tag{2}$$

and eliminate the flux term, it results to the following Cattaneo heat transfer equation

$$\rho c \frac{\partial T}{\partial t} + \rho c \tau \frac{\partial^2 T}{\partial t^2} = k\nabla^2 T + \tau \frac{\partial q_{gen}}{\partial t} + q_{gen} \tag{3}$$

Since Eq. (3) is damped wave equation, it is also called thermal wave model because it is purely propagative in nature. Cattaneo model has been studied both experimentally and theoretically in different fields of engineering which are, but not limited to, microscale heat transfer [4, 5], laser interaction with metals [6, 7], bioheat transfer [8-11], heat transfer in porous materials [12], fluid mechanics [13], and heat transfer in solid reactanats [14].

Different ranges of values for the relaxation time τ in Cattaneo model have been given in different heat transfer processes and materials. Table 1 lists the values of relaxation time for some typical materials. As can be seen from Table 1, τ for nonhomogeneous materials differs greatly to that of metals and semi- and super conductors. The possible explanation for this difference is given in [12].

Another well-known and extensively used non-Fourier model is based on phase-lagging or delayed constitutive models. The simplest model of this kind is *single-phase-lag* model which is based on the assumption of the existence of a delay between the heat flux and its temperature gradient in the following form:

$$q(\vec{r},t+\tau_q)=-k\nabla T(\vec{r},t) \quad (4)$$

where τ is phase-lag in time. Expanding Eq. (4) up to first order by Taylor series expansion (assuming τ is small compared with t) results to

$$q(\vec{r},t+\tau)=q(\vec{r},t)+\tau_q\frac{\partial q(\vec{r},t)}{\partial t}=-k\nabla T(\vec{r},t) \quad (5)$$

Equation (5) is identical to Eq. (1) but has been derived through phase-lagging concept. Another phase-lagging based model is Dual-Phase Lag (DPL) model in which there exists another time lag in the constitutive model in the following form

$$q(\vec{r},t+\tau_q)=-k\nabla T(\vec{r},t+\tau_t) \quad (6)$$

where τ_t is called temperature gradient time lag. τ_q has the same explanation as in Eq. (1) and is called heat flux time lag. Based on microscale approach, τ_t is responsible for the time lag resulting from the microstructural interaction. However, DPL model is widely used not only in modeling heat transfer processes in microscale but also in modeling anomalous behaviors of thermal transport observed in macroscale. This anomalous behavior which is for some cases wave-like and for other cases sub- or super-diffusive, cannot be captured through Fourier or thermal wave model which are purely diffusive and propagative in nature, respectively. Some fields of application of DPL model are modeling heat transfer in microscale heat transfer [5], porous medium [15] and bioheat transfer [16, 17].

3. The fractional heat conduction constitutive model

As observed in the previous section, in both thermal wave and DPL models, expanded heat conduction equation can be obtained by inserting (at least) first-order Taylor series expansion of non Fourier constitutive equations (single- or dual-phase lag models) into the heat balance relation i.e. eliminating heat flux between two equations.

Recently, Odibat et al. [18] introduced a fractional Taylor series expansion according to which we can expand single-phase lag constitutive model (Eq. (4)) in the following form:

$$q(\vec{r},t+\tau)=q(\vec{r},t)+\frac{\tau^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}q(\vec{r},t)}{\partial t^{\alpha}}=-k\nabla T(\vec{r},t)\ ,\ 0<\alpha\le 1 \quad (7)$$

where α is fractional order of differentiation. This kind of fractional Taylor series formula has been recently used to derive the fractional-order conservation of mass equation and it has been shown that fractional Taylor series of mass flux is able to exactly represent non-linear flux in a control volume using only the first two terms as long as the flux can be written as a power law function with the same exponent as the order of differentiation α [19].

Returning to Eq. (7), it is obvious that the dimension of time lag τ remains “second” as in the two previous non-Fourier models. From Eq. (7), it is obvious that when $\tau=0$, no thermal lag, the fractional non-Fourier model reduces to Fourier model (pure diffusion) and for $\tau\neq 0, \alpha=1$, it reduces to thermal wave model and traditional integer-order Taylor series is also recovered. Finally for $\tau\neq 0, 0<\alpha<1$, because of the non-local property of fractional derivative introduced through the memory kernel (memory integral) existing in the definition of fractional derivative, it models the intermediate processes between thermal wave and pure diffusion (parabolic) which cannot be captured through integer-order Taylor series expansion of single-phase lag model i.e. Eq. (1). This feature is carried out in DPL model by adjusting the values of τ_q and τ_t.

Several definitions of a fractional derivative have been proposed, for example the definitions of Riemann-Liouville, Grünwald-Letnikov and Caputo [3]. From the historical point of view, Riemann-Liouville definition seems to be the most important definition among the others, because many of the later achievements of fractional calculus have come from this definition. However, the fractional Taylor series is valid only with Caputo definition [18]. Caputo definition enables us to use initial/boundary conditions (in this study initial conditions since fractional derivative has been applied in time domain) of practical type in the formulation. Also it has the advantage of employing the classical Laplace transform for fractional derivatives which is very useful in deriving the analytic solution as is done in the next section. In other words, Riemann-Liouville definition is more preferred in pure mathematics, but not in applied and engineering problems which involve standard definitions for initial/boundary conditions. For this reason Caputo definition is preferred in modeling practical problems. On the other hand, Grünwald-Letnikov definition serves as an efficient definition for numerical approximation of a fractional derivative based on Riemann-Liouville definition. Since these two definitions are equivalent for a wide class of functions.
These three definitions are defined as [3]:

$$\text{Riemann-Liouville}: \ {}_0D^{\alpha} f(t) \equiv \frac{d^{\alpha} f}{dt^{\alpha}} \equiv \frac{1}{\Gamma(m-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \quad m-1<\alpha<m,\ m \in N, \tag{8-a}$$

$$\text{Grünwald-Letnikov}: \ D^{\alpha} f(t) \equiv \frac{d^{\alpha} f}{dt^{\alpha}} \equiv \frac{1}{\Delta t^{\alpha}} \sum_{v=0}^{n} (-1)^v \binom{\alpha}{v} f(t_{n-v}) \ , \tag{8-b}$$

$$\text{Caputo}: \quad D_*^{\alpha} f(t) \equiv \frac{d^{\alpha} f}{dt^{\alpha}} \equiv \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \quad m-1<\alpha<m,\ m \in N, \tag{8-c}$$

Since the generalized Taylor formula is based on Caputo definition, Eq. (7) is consequently based on Caputo definition Eq. (8-c) with the following Laplace transform

$$\mathcal{L}\left\{\frac{d^{\alpha} f}{dt^{\alpha}}\right\} \equiv s^{\alpha} \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0), \tag{9}$$

Inserting Eq. (7) into energy balance equation (2) and eliminating the flux term results in the following generalized fractional heat conduction equation:

$$\rho c \frac{\partial T}{\partial t} + \frac{\rho c \tau^{\alpha}}{\Gamma(1+\alpha)} \frac{\partial^{(\alpha+1)} T}{\partial t^{(\alpha+1)}} = k \nabla^2 T + \frac{\tau^{\alpha}}{\Gamma(1+\alpha)} \frac{\partial^{\alpha} q_{gen}}{\partial t^{\alpha}} + q_{gen} \tag{10}$$

4. Illustrative example

In order to understand the role of fractional order α in Eq. (7), two illustrative examples are chosen to be solved analytically through finite sine-Fourier and Laplace transforms. The details of the procedure are also presented, for they are thought to be useful in the context of fractional calculus.
Consider a one dimensional fractional non-Fourier heat conduction problem in a finite slab with no heat generation

$$\rho c \frac{\partial T}{\partial t} + \frac{\rho c \tau^{\alpha}}{\Gamma(\alpha+1)} \frac{\partial^{(\alpha+1)} T}{\partial t^{(\alpha+1)}} = k \frac{\partial^2 T}{\partial x^2} \tag{11}$$

under homogeneous initial conditions

$$\begin{aligned} t = 0 &: T = 0, \\ t = 0 &: \frac{\partial T}{\partial t} = 0, \end{aligned} \tag{12}$$

and the following constant step heating boundary conditions

$$x = 0 : \; T = T_0 = constant, \quad x = L : \; T = 0, \tag{13}$$

Taking the finite sine-Fourier transform of Eq. (11) and applying boundary conditions (13), we obtain

$$\frac{d\bar{T}}{dt} + \frac{\tau^{\alpha}}{\Gamma(\alpha+1)} \frac{d^{(\alpha+1)}\bar{T}}{dt^{(\alpha+1)}} = \mathcal{D}\left(-\xi_n^2 \bar{T} + \xi_n T_0\right), \tag{14}$$

where $\mathcal{D} = k/\rho c$ is thermal diffusivity, $\xi_n = n\pi/L$ and

$$\bar{T} = \bar{T}(n,t) = \int_0^L T(x,t) \sin(\xi_n x)\,\mathrm{d}x \;, \tag{15}$$

is the finite sine-Fourier transform of $T(x,t)$. Taking the Laplace transform of Eq. (14) and using Eq. (9) and initial conditions (12), we obtain

$$\bar{T}^* = \frac{\mathcal{D}\xi_n T_0}{s\left(s + \frac{\tau^{\alpha}}{\Gamma(\alpha+1)} s^{\alpha+1} + \mathcal{D}\xi_n^2\right)}, \tag{16}$$

where asterisk denotes Laplace transform. Using partial fraction, Eq. (16) takes the form

$$\bar{T}^* = \frac{T_0}{\xi_n}\left[\frac{1}{s} - \frac{1 + \frac{\tau^{\alpha}}{\Gamma(\alpha+1)} s^{\alpha}}{s + \frac{\tau^{\alpha}}{\Gamma(\alpha+1)} s^{\alpha+1} + \mathcal{D}\xi_n^2}\right], \tag{17}$$

Eq. (17) can be simplified as

$$\bar{T}^* = \frac{T_0}{\xi_n}\frac{1}{s} - \frac{T_0}{\xi_n}\left[s^{-1}\frac{\varsigma}{1+\varsigma} + \frac{\tau^{\alpha}}{\Gamma(\alpha+1)} s^{\alpha}\frac{\varsigma}{1+\varsigma}\right], \tag{18}$$

where

$$\varsigma = \frac{\tau^{-\alpha}\Gamma(\alpha+1)s}{s^{\alpha+1} + \mathcal{D}\xi_n^2 \tau^{-\alpha}\Gamma(\alpha+1)}, \tag{19}$$

We can rewrite Eq. (18) in the following form

$$\begin{aligned}\bar{T}^* = {} & \frac{T_0}{\xi_n}\frac{1}{s} - \frac{T_0}{\xi_n}\sum_{k=0}^{\infty}(-1)^k \tau^{-\alpha(k+1)}\left(\Gamma(\alpha+1)\right)^{k+1} \frac{s^k}{\left(s^{\alpha+1} + \mathcal{D}\xi_n^2\tau^{-\alpha}\Gamma(\alpha+1)\right)^{k+1}} \\ & - \frac{T_0}{\xi_n}\sum_{k=0}^{\infty}(-1)^k \tau^{-\alpha k}\left(\Gamma(\alpha+1)\right)^{k} \frac{s^{k+\alpha}}{\left(s^{\alpha+1} + \mathcal{D}\xi_n^2\tau^{-\alpha}\Gamma(\alpha+1)\right)^{k+1}},\end{aligned} \tag{20}$$

Considering the Laplace transform of Mittag-Leffler function [3]

$$\int_0^{\infty} e^{-st}\left(t^{\alpha n+\beta-1}\mathrm{E}_{\alpha,\beta}^{(n)}\left(\pm a t^{\alpha}\right)\right)\mathrm{d}t = \frac{n!s^{\alpha-\beta}}{\left(s^{\alpha} \mp a\right)^{n+1}}, \tag{21}$$

where $\mathrm{E}_{\alpha,\beta}^{n}(z)$ is the two-parameter Mittag-Leffler function which can be said as a counterpart of exponential function in fractional calculus

$$\mathrm{E}_{\alpha,\beta}^{(n)}(z) \equiv \frac{d^n}{dz^n}\mathrm{E}_{\alpha,\beta}(z) = \sum_{j=0}^{\infty}\frac{(j+n)!z^j}{j!\Gamma(\alpha j + \alpha n + \beta)}, \tag{22}$$

then, term-by-term Laplace inversion of (20) gives

$$\bar{T} = \frac{T_0}{\xi_n} - \frac{T_0}{\xi_n}\sum_{k=0}^{\infty}\frac{(-1)^k \tau^{-\alpha(k+1)} t^{\alpha(k+1)} \left(\Gamma(\alpha+1)\right)^{k+1}}{k!} \mathrm{E}^{(k)}_{\alpha+1,\alpha+1-k}\left(-\mathcal{D}\xi_n^2\tau^{-\alpha}\Gamma(\alpha+1)t^{\alpha+1}\right)$$
$$-\frac{T_0}{\xi_n}\sum_{k=0}^{\infty}\frac{(-1)^k \tau^{-\alpha k} t^{\alpha k}\left(\Gamma(\alpha+1)\right)^k}{k!}\mathrm{E}^{(k)}_{\alpha+1,1-k}\left(-\mathcal{D}\xi_n^2\tau^{-\alpha}\Gamma(\alpha+1)t^{\alpha+1}\right), \tag{23}$$

Taking the inverse finite sine-Fourier transform of Eq. (23) gives the exact explicit solution of temperature distribution as

$$T = \frac{2}{L}\sum_{n=1}^{\infty}\left[\frac{T_0}{\xi_n} - \frac{T_0}{\xi_n}\sum_{k=0}^{\infty}\frac{(-1)^k \tau^{-\alpha(k+1)}\left(\Gamma(\alpha+1)\right)^{k+1} t^{\alpha(k+1)}}{k!}\mathrm{E}^{(k)}_{\alpha+1,\alpha+1-k}\left(-\mathcal{D}\xi_n^2\tau^{-\alpha}\Gamma(\alpha+1)t^{\alpha+1}\right)\right.$$
$$\left.-\frac{T_0}{\xi_n}\sum_{k=0}^{\infty}\frac{(-1)^k\tau^{-\alpha k}\left(\Gamma(\alpha+1)\right)^k t^{\alpha k}}{k!}\mathrm{E}^{(k)}_{\alpha+1,1-k}\left(-\mathcal{D}\xi_n^2\tau^{-\alpha}\Gamma(\alpha+1)t^{\alpha+1}\right)\right]\sin(\xi_n x), \tag{24}$$

If τ=0, the result is reduced to classical Fourier model. In this case, the solution (24) becomes

$$T = \frac{2}{L}\sum_{n=1}^{\infty}\frac{T_0}{\xi_n}\left[1 - \exp(-\mathcal{D}\xi_n^2)\right]\sin(\xi_n x), \tag{25}$$

If α=1 and $\tau\neq0$, then solution is reduced to thermal wave mode. From Eq. (16) we obtain

$$T = \frac{2}{L}\sum_{n=1}^{\infty}\frac{T_0}{\xi_n}\left[1 + \frac{s_2\exp(s_1 t) - s_1\exp(s_2 t)}{s_1 - s_2}\right]\sin(\xi_n x), \tag{26}$$

where

$$s_{1,2} = \frac{-1 \pm \sqrt{1 - 4\mathcal{D}\xi_n^2\tau}}{2\tau}, \tag{27}$$

The dimensionless form of the final solution (24) can be obtained by introducing the following dimensionless parameters in the solution:

$$\varepsilon = \frac{x}{L},\ \eta = n\pi,\ \delta = \frac{\mathcal{D}}{L^2}\tau,\ \kappa = \frac{\mathcal{D}}{L^2}t,\ \theta = \frac{T}{T_0} \tag{28}$$

Then the compact dimensionless form of the solution becomes:

$$\theta = 2\sum_{n=1}^{\infty}\left[\frac{1}{\eta} - \frac{1}{\eta}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\kappa}{\delta}\right)^{\alpha(k+1)}\left(\Gamma(\alpha+1)\right)^{k+1}\mathrm{E}^{(k)}_{\alpha+1,\alpha+1-k}\left(-\frac{\kappa^{\alpha+1}}{\delta^{\alpha}}\eta^2\Gamma(\alpha+1)\right)\right.$$
$$\left.-\frac{1}{\eta}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\kappa}{\delta}\right)^{\alpha k}\left(\Gamma(\alpha+1)\right)^k\mathrm{E}^{(k)}_{\alpha+1,1-k}\left(-\frac{\kappa^{\alpha+1}}{\delta^{\alpha}}\eta^2\Gamma(\alpha+1)\right)\right]\sin(\varepsilon\eta), \tag{29}$$

As can be seen from Eq. (29), multiple summations appear in the solution which is very common in finding the analytic solution of fractional partial differential equations. The problem which arises next is the slow convergence of the solution and the difficulty of numerical processing of Eq. (29). Therefore, Laplace transform and Riemann-sum approximation for Laplace inversion have been used as an alternative approach. In this technique, we first take the Laplace transform of the governing partial differential equation and then, by applying the boundary conditions, solve it for finding $\Theta(s,\varepsilon)$ which is the Laplace transform of the dependent variable $\theta(\kappa,\varepsilon)$ of the problem. Then for Laplace inversion, we use Riemann sum approximation for Bromwich contour integral in the following form:

$$\theta(\kappa,\varepsilon)=\frac{e^{\gamma\kappa}}{\kappa}\left[\frac{1}{2}\Theta(s=\gamma,\varepsilon)+\mathrm{Re}\sum_{n=1}^{N}\Theta\left(s=\gamma+\frac{in\pi}{\kappa},\varepsilon\right)(-1)^{n}\right] \tag{30}$$

where the value of γ is equal to $(2+e)/\kappa$ for faster convergence of the numerical summation and Re returns to the real part of the summation. In Eq. (30), e is the neperian number.

Figures 1-6 show the temperature distribution for the case of step heating for α in decreasing order. As α changes from 1 (thermal wave solution) in Fig. 1 to zero (Fourier solution) in Fig. 6, the sharpness of the propagation of step boundary perturbation into the medium decreases; in other words, hyperbolicity decreases and diffusivity becomes more and more dominant as α decreases. This feature can be more observable if we find the temperature distribution for the case of a pulse heating. In this case, initial conditions are the same as step heating but boundary conditions are defined as follows

$$\varepsilon=0:\ \begin{cases}\theta=1, & 0\le\kappa<0.02\\ \theta=0, & 0.02\le\kappa\end{cases}$$
$$\varepsilon=1:\ \theta=0, \tag{31}$$

The procedure in finding the analytic solution for pulse heating is similar to that of step heating presented above. Therefore, it has not been included for the sake of preserving space. In Figs. 7-12, temperature distribution for the case of pulse-heating has been presented. Two important features are readily observable in these results: first: as α decreases from 1 in Fig. 7 to zero in Fig. 12, *heat effected zone* in the medium resulted from boundary thermal perturbation increases. This feature can be observed from the *wider dome* (solid line indicated in figures 7-12) that is formed as α decreases through figures 7-12. Second feature returns to the change in the concept of *memory* of modeling as α reduces from thermal wave case in Fig. 7 to Fourier (purely diffusive nature) in Fig. 12. Again, taking a closer look at dome in these figures we observe that the height of the dome reduces as α decreases. Keeping in mind that the boundary perturbation is a pulse, we conclude that as α decreases, the modeling loses its *memory* from the thermal perturbation applied at x=0.

It should be mentioned that in DPL model, as mentioned in previous section, in order to capture the *wave-like* behavior of thermal response, some diffusion is injected through adjusting the value of τ_t in order to affect the purely wave behavior of τ_q. In fractional Cattaneo model presented in this study and as observed in graphical results, this wave-like behavior is captured through lowering the hyperbolicity of the modeling through fractional (non-integer) order of differentiation α introduced in Eq. (7).

5 Conclusions

By applying the generalized Taylor's formula on single-phase lag equation, a macroscopic fractional heat conduction constitutive model is derived. Based on the resulting fractional model and using the Laplace and finite sine-Fourier transforms, exact solutions for step and pulse heating are obtained and shown graphically. It has been shown that the resulting fractional constitutive model can capture the intermediate processes between pure diffusion and thermal wave through changing the order of fractional differentiation α. In other words, fractional constitutive model allows the *interpolation* between pure diffusion and thermal wave nature of heat transfer. Considering the great capabilities of fractional derivatives in modeling dynamics of complex systems and processes, it seems promising to use fractional calculus to model anomalous heat transfer processes more accurately in future.

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Nomenclature

C: Specific heat at constant pressure
C_t: Thermal wave speed
D: Thermal diffusivity
$E_{\alpha,\beta}$: Two parameter Mittag-Leffler function
k: Thermal conductivity
q: Heat flux
q_{gen}: Heat source term
t: Time
T: Temperature

T_0: Constant temperature used as boundary condition

$\bar{T}$: Finite sine-Fourier transform of T

Greek symbols

α : Fractional (non-integer) order of time derivative (fractionality)
δ : Dimensionless time lag
ε : Dimensionless space variable
κ : Dimensionless time
θ : Dimensionless temperature
ρ : Density
τ : Time lag
τ_q : Heat flux time lag
τ_t : Temperature gradient time lag

Table 1 values of relaxation time τ [18]

Materials	τ (in seconds) as function of T cryogenic	room	high
Metals			
Aluminum	10^{-11}-10^{-6}	10^{-14}-10^{-11}	$<10^{-14}$
Tantalium	10^{-8}-10^{-6}	10^{-13}-10^{-8}	$<10^{-13}$
Superconductors			
Tantalium	$\approx 10^{-8}$		
Niobium	$\approx 10^{-8}$		
YBaCuO	$\approx 10^{-10}$	$\approx 10^{-12}$	
Semiconductors			
Gallium Arsenide	10^{-10}-10^{-7}	10^{-13}-10^{-10}	$<10^{-13}$
Organic Materials			
Tissue	10-1000	1-100	
Porous materials			
Sand (187 μm, 42% porosity)		≈20	
Glass (206 μm, 36% porosity)		≈10	

Figures

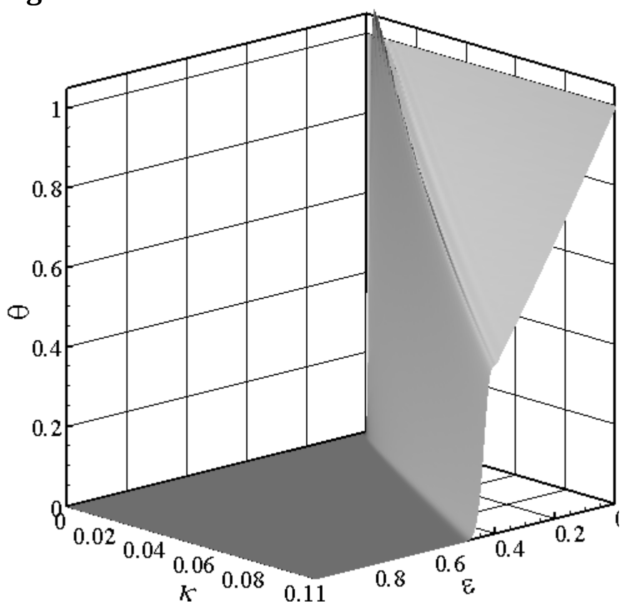

Figure 1 Thermal response for the case of step heating, α=1.0, δ=0.05

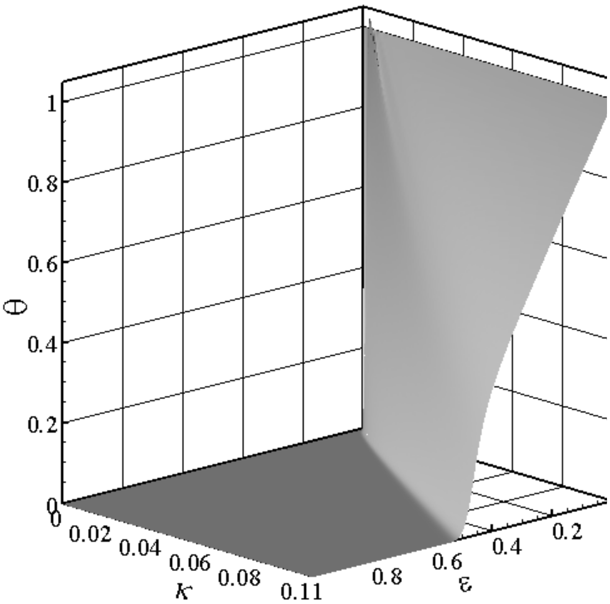

Figure 2 Thermal response for the case of step heating, α=0.95, δ=0.05

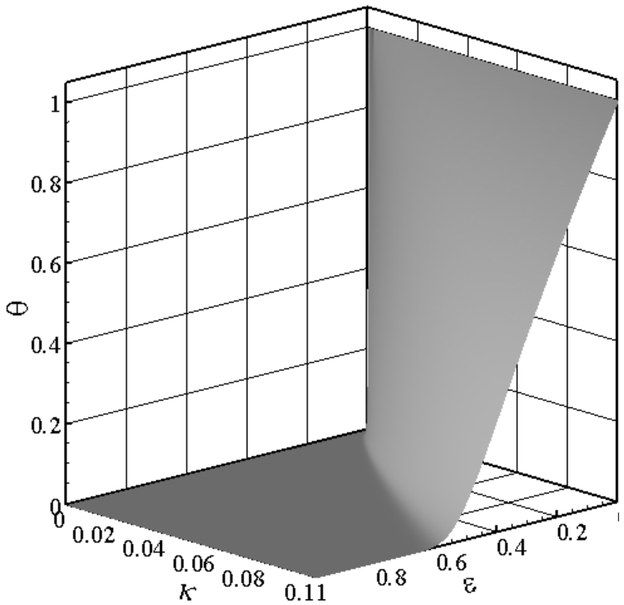

Figure 3 Thermal response for the case of step heating, α=0.8, δ=0.05

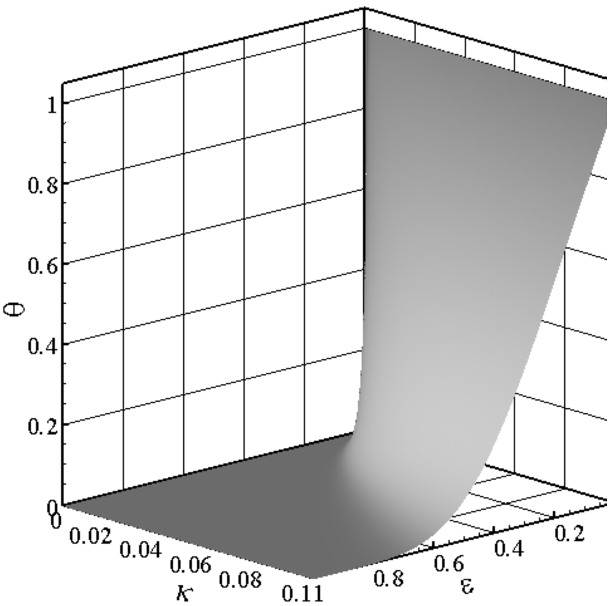

Figure 4 Thermal response for the case of step heating, α=0.5, δ=0.05

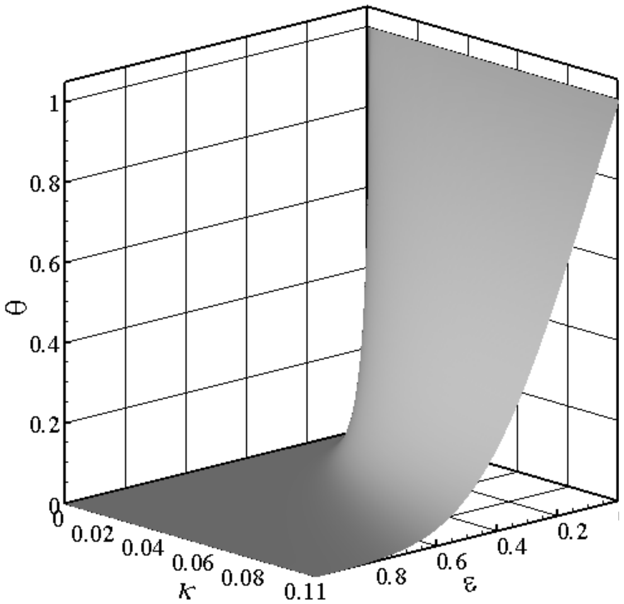

Figure 5 Thermal response for the case of step heating, α=0.3, δ=0.05

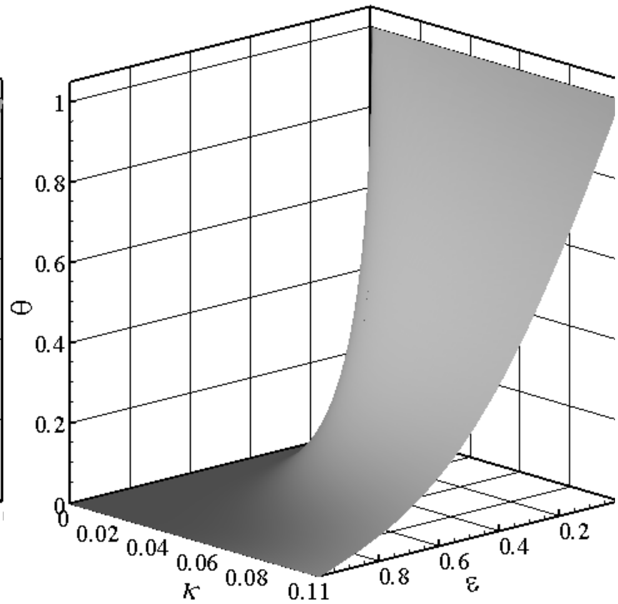

Figure 6 Thermal response for the case of step heating, α=0.0, δ=0.05

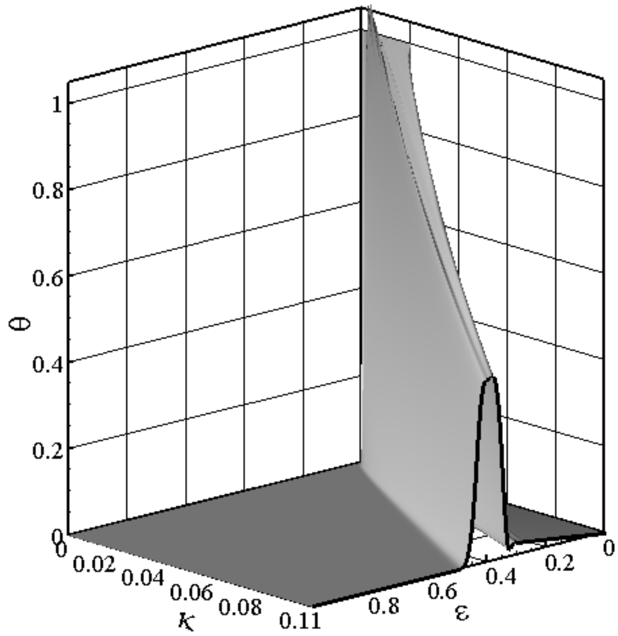

Figure 7 Thermal response for the case of pulse heating, α=1.0, δ=0.05

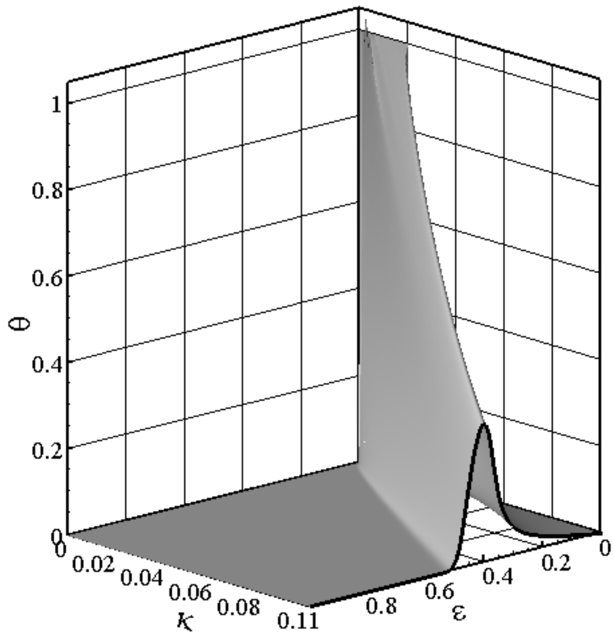

Figure 8 Thermal response for the case of pulse heating, α=0.95, δ=0.05

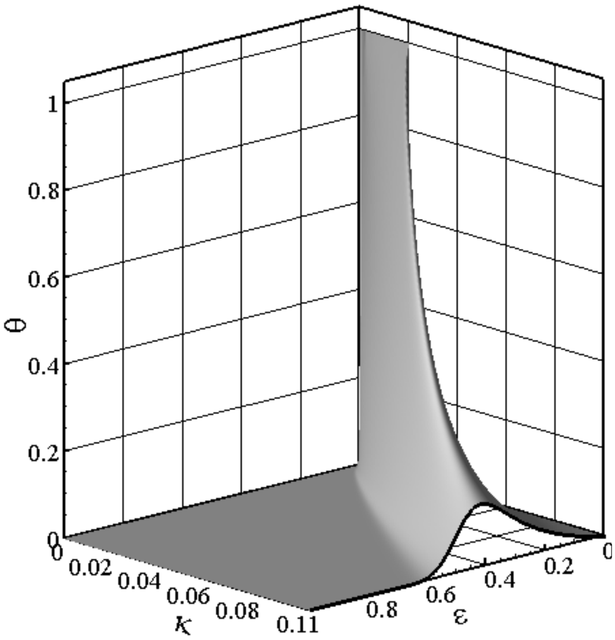

Figure 9 Thermal response for the case of pulse heating, α=0.8, δ=0.05

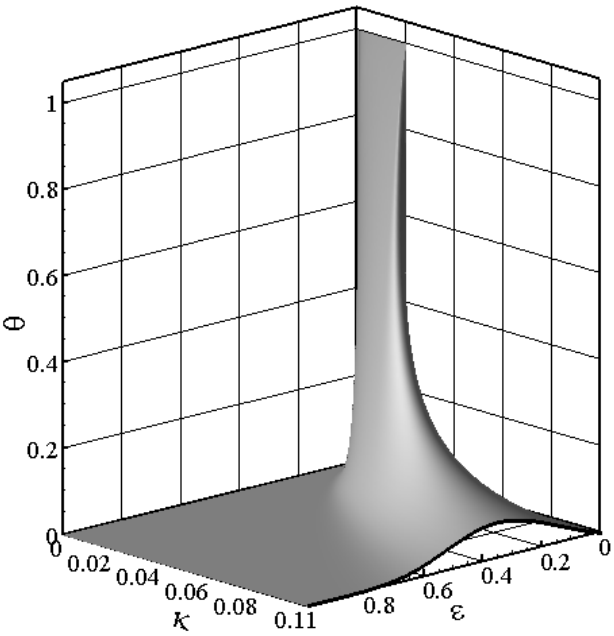

Figure 10 Thermal response for the case of pulse heating, α=0.5, δ=0.05

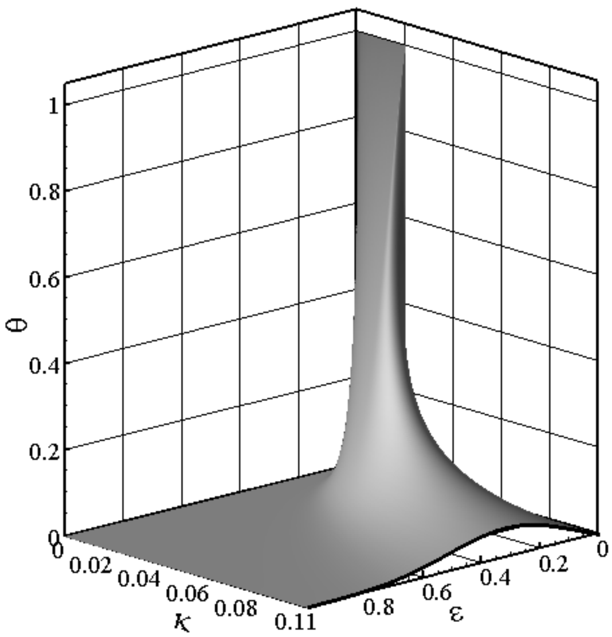

Figure 11 Thermal response for the case of pulse heating, α=0.3, δ=0.05

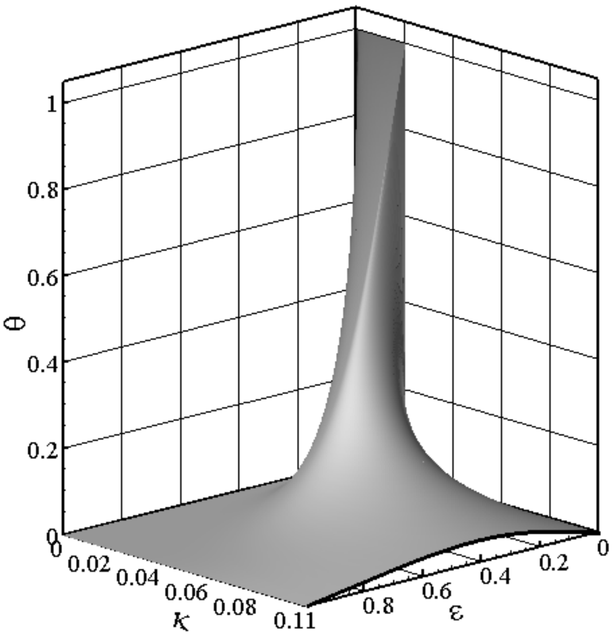

Figure 12 Thermal response for the case of pulse heating, α=0.0, δ=0.05

چکیده

در این مقاله، با اعمال بسط سری تیلور کسری بر روی مدل ساختاری تاخیر زمانی منفرد، یک معادله ساختاری کسری انتقال حرارت معرفی می‌گردد. با ترکیب معادله ساختاری کسری معرفی شده با معادله بالانس انرژی، معادله کلی انتقال حرارت بدست می‌آید. این معادله انتقال حرارت بدست آمده قادر است که رفتارهای میانی انتقال حرارت بین مدل‌های کلاسیک فوریه و موج حرارتی را مدل نماید. همچنین، با استفاده از تبدیل لاپلاس و تبدیل فوریه سینوسی، جواب دقیق مسئله انتقال حرارت کسری برای تحریک پله و پالس ارائه گردیده است.