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# Thermal Buckling of Functionally Graded Beams

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*In this article, thermal stability of beams made of functionally graded material (FGM) is considered. The derivations of equations are based on the one-dimensional theory of elasticity. The material properties vary continuously through the thickness direction. Tanigawa's model for the variation of Poisson's ratio, the modulus of shear stress, and the coefficient of thermal expansion is considered. The equilibrium and stability equations for the functionally graded beam under thermal loading are derived using the variational and force summation methods. A beam containing six different types of boundary conditions is considered and closed form solutions for the critical normalized thermal buckling loads related to the uniform temperature rise and axial temperature difference are obtained. The results are reduced to the buckling formula of beams made of pure isotropic materials.*



## 1 Introduction

In recent years, functionally graded materials (FGMs) have gained considerable importance in design of structures under extremely high temperature environments, such as chemical plants. FGMs are also considered as potential structural material designed for use in thermal barrier coatings in different structural applications.

A survey of the literature reveals that the problem of thermal buckling in straight and curved beams and circular rings subjected to temperature distribution of arbitrary variation has not been treated in a general form. Roark and Young [1] presented solutions for curved beams of various boundary conditions under the action of uniform temperature distribution along the span, but varying linearly through the thickness of the beam based on Castigliano's theory. Forray [2] gave only the stresses in closed circular rings subjected to a temperature distribution of general trigonometric variation, and then furnished design equations for stresses in closed rings under some special temperature distributions in another article [3]. Parkus [4] discussed very briefly the problem of slightly curved bars. For thin rings (ratio greater than about 10), the Winkler curved beam theory [5], which accounts for the hyperbolic distribution of strain, is not only too cumbersome but potentially capable of generating non-equilibrium stress resultants as thin approximations are introduced [6]. The analysis of rings and curved beams subjected to out of plane loads reported by Fettahlioglu and Tabi [7] and subjected to in plane loads reported

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by Fettahlioglu and Tabi [8], form a complete treatment. The stress analysis of curved beams of any specific set of constraints under the action of loads of any type can be performed by these later references. Cheng and Hoff [9] have presented the analysis based on inextensional deformations of rings under the static surface loads.

Formula for the elastic buckling of beams may be applied to conditions under which the proportional limit exceeds. This is when a reduced value of the modulus of elasticity corresponding to the actual stress is used [10]. Carter and Gere [11] presented the graphs of buckling coefficients for beams with single tapers for various end conditions, cross sections, and degrees of taper. Culver and Preg [12] investigated and tabulated the buckling coefficients for singly tapered beam-columns in which the effect of torsion, including warping restraint, is considered. This is the case where the load is through the end moments in the stiffer principal plane. Kitipornchai and Trahair [13] described the lateral stability of singly tapered cantilever and doubly tapered simple I-beams, including the effect of warping restraint, where the experimental results are compared with the numerical solutions. Morrison [14] considers the effect of lateral restraint of the tensile flange of a beam under lateral buckling. Massey and McGuire [15] presented the graphs of buckling coefficients for both stepped and tapered cantilever beams, where good agreements with experiments are reported. Fowler [16] has presented tables of lateral stability constants for laminated timber beams. Clark and Hill [17] have derived a general expression for the lateral stability of unsymmetrical I-beams with boundary conditions based on both bending and warping supports, where the tables of coefficients and nomographs are presented. Anderson and Trahair [18] have presented the tabulated lateral buckling coefficients for the uniformly loaded and end-loaded cantilevers and center and uniformly loaded simply supported beams having unsymmetric I-beam cross sections. Comparisons are made with extensive tests on cantilever beams. Roorda [19] discusses the extension of initial imperfections into the nonlinear range of beam buckling. Burgreen and Manitt [20] and Burgreen and Regal [21] presented the analysis of bimetallic beams and have pointed out some of the difficulties in predicting the snap-through instability of these beams under changes in temperature. Austin [22] have tabulated the in-plane buckling coefficients for circular, parabolic, and catenary arches for pinned and fixed ends and the three-hinged case. He considers cases where the cross section varies with the position in the span and the case of a uniform cross section. Uniform loads, unsymmetric distributed loads, and concentrated center loads are considered, and the stiffening effect of tying the arch to the girder with columns is also evaluated. A thin ring shrunk by cooling and inserted into a circular cavity usually yield before buckling unless the radius/thickness ratio is very large and the elastic-limit stress is high. Chicurel [23] derived approximate solutions to this problem when the effect of friction is considered. He suggests a conservative expression for the no-friction condition.

Buckling analysis of functionally graded structures are also reported in the literature. Birman [24] studied the buckling problem of functionally graded composite rectangular plate subjected to the uniaxial compression. The stabilization of a functionally graded cylindrical shell under the axial harmonic loading is investigated by Ng et al. [25]. Shahsiah and Eslami presented the thermal buckling of cylindrical shell made of functionally graded material based on the first order shell theory and the Donnell and improved Donnell equations [26-27]. The buckling analysis of circular functionally graded plates is given by Najafizadeh and Eslami [28]. Javaheri and Eslami presented the thermal and mechanical buckling of rectangular functionally graded plates based on the first and higher order plate theories [29-32]. The thermal and mechanical buckling of functionally graded rings and curved beams are reported by Shahsiah and Eslami [33] and Shafii et al. [34].

In this article, the normalized functions proportional to the thermal buckling loads for thin beams made of functionally graded material are derived. Beam is under uniform temperature rise and axial temperature difference. Beam with six types of boundary conditions is assumed. The expressions for the critical thermal buckling loads are obtained analytically and are given by approximate solutions. It is further assumed that the beam is thin enough so that shift of the neutral axis is negligible.

## 2 Variational formulation

Consider a thin straight beam made of functionally graded material (FGM). The length of the beam is $L$, its width is $b$, and its cross sectional height is $h$. The coordinate system $x - z$ is considered such that the $x$-axis is along the length of the beam on the neutral axis (centerline) and the $z$-axis is along the height of the cross section. The beam material is graded across the $z$-axis with variation profile proposed by Tanigawa [35] as

$$\alpha(z) = \alpha_0(\frac{z}{a} + 1)^k \tag{1}$$

$$G(z) = G_0(\frac{z}{a} + 1)^m \tag{2}$$

$$\nu = \frac{1}{m+2} \tag{3}$$

where $\alpha(z)$, $\alpha_0$, $G(z)$, and $G_0$ are the thermal expansion coefficient varying across the height of the beam, the coefficient of thermal expansion on the beam's neutral plane at $z = 0$, the shear modulus varying across the height of the beam, and the shear modulus on the beam's neutral plane at $z = 0$, respectively. Here, $\nu$ is Poisson's ratio and $m$ and $k$ are the arbitrary power indices. Term $a$ is a characteristic length defined subsequently and used to normalize the physical quantities.

The elastic total potential energy of the beam made of functionally graded material under thermal load is given as

$$V = U_m + U_b + Y_T \tag{4}$$

where $V$ is the total potential energy, $U_m$ is the membrane strain energy, $U_b$ is the bending strain energy, and $U_T$ is the thermal strain energy. The membrane strain energy is

$$U_m = \frac{1}{2}\int_v \sigma_x \epsilon_x dv \tag{5}$$

where $\sigma_x$, $\epsilon_x$, and $v$ are the axial normal stress, axial normal strain, and total volume of the functionally graded beam, respectively.

Using Hooke's law and the mechanical properties defined by Eqs. (1), (2), and (3), the membrane strain energy of the thin beam of width $b$ is

$$\begin{aligned}
U_m &= \frac{1}{2}\int_x\int_y\int_z \sigma_x \epsilon_x dxdydz = \frac{1}{2}\int_0^L\int_0^b\int_{-h/2}^{+h/2} E(z)\epsilon_x^2 dxdydz \\
&= (1+\nu)\int_0^L\int_0^b\int_{-h/2}^{+h/2} G_0(\frac{z}{a}+1)^m \epsilon_x^2 dxdydz \\
&= \frac{m+3}{m+2}G_0 b\int_0^L \epsilon_x^2 dx \int_{-h/2}^{+h/2}(\frac{z}{a}+1)^m dz \\
&= \frac{(m+3)}{(m+2)(m+1)}G_0 ab\left[(1+\frac{h}{2a})^{m+1} - (1-\frac{h}{2a})^{m+1}\right]\int_0^L \epsilon_x^2 dx
\end{aligned} \tag{6}$$

where $E(z)$ is the elastic modulus varying continuously across the beam's height. The beam is assumed to be thin enough such that the shift of neutral axis of the beam is assumed to be negligible. The bending strain energy of the functionally graded beam is

$$U_b = \frac{1}{2} \int_v \frac{E(z)I_{yy}}{A}\ K_y^2 dv = (1+\nu) \int_v \frac{G(z)I_{yy}}{A}\ K_y^2 dv \tag{7}$$

where $I_{yy}$ and $A$ are the second moment of inertia of the cross section about the $y$-axis, and the cross sectional area of the beam, respectively, and $K_y$ is the curvature of the beam in $x-z$ plane. Substituting Eq. (2) in Eq. (7) results the bending strain energy as

$$\begin{aligned} U_b &= \frac{m+3}{m+2} \int_0^L \int_0^b \int_{-h/2}^{+h/2} \frac{G_0 I_{yy} K_y^2}{A}\ (\frac{z}{a}+1)^m dxdydz \\ &= \frac{m+3}{m+2} \int_0^L \int_0^b \int_{-h/2}^{+h/2} \frac{G_0 I_{yy} M(x)^2}{AE(z)^2 I_{yy}^2}\ (\frac{z}{a}+1)^m dxdydz \\ &= \frac{b(m+2)}{4G_0 I_{yy} A(m+3)} \int_0^L M(x)^2 dx \int_{-h/2}^{+h/2} (\frac{z}{a}+1)^{-m} dz \\ &= \frac{a(m+2)}{4G_0 I_{yy} h(m+3)(1-m)} \left[(1+\frac{h}{2a})^{1-m} - (1-\frac{h}{2a})^{1-m}\right] \int_0^L M(x)^2 dx \end{aligned} \tag{8}$$

where $M(x)$ is the bending moment in $x$-direction. The thermal strain energy is

$$U_T = -\frac{1}{2} \int_v \sigma_x \epsilon_T dv = -\frac{1}{2} \int_v \sigma_x \alpha(z) \Delta T dv \tag{9}$$

where $\Delta T$ and $\epsilon_T$ are the temperature difference in the beam and thermal strain, respectively. Substituting Eqs. (1), (2), and (3) in Eq. (9) gives

$$\begin{aligned} U_T &= -(1+\nu) \int_0^L \int_0^b \int_{-h/2}^{+h/2} G(z)\alpha(z)\Delta T \epsilon_x dxdydz \\ &= -G_0\alpha_0 b\ \frac{m+3}{m+2} \int_0^L \Delta T \epsilon_x dx \int_{-h/2}^{+h/2} (\frac{z}{a}+1)^{k+m} dz \\ &= -G_0\alpha_0 ab\ \frac{m+3}{(m+2)(k+m+1)} \left[(1+\frac{h}{2a})^{k+m+1}\right. \\ &\quad \left. -(1-\frac{h}{2a})^{k+m+1}\right] \int_0^L \Delta T \epsilon_x dx \end{aligned} \tag{10}$$

The axial strain $\epsilon_x$ and the bending moment $M(x)$ are related to the axial displacement $u$ and the lateral displacement $w$ as [2]

$$\epsilon_x = \frac{du}{dx} + \frac{1}{2}\ (\frac{dw}{dx})^2 \tag{11}$$

$$M(x) = E(z) I \frac{d^2 w}{dx^2} \tag{12}$$

Substituting Eqs. (11) and (12) in Eqs. (6), (8), and (10) and finally in Eq. (4), result into the total potential energy in terms of the displacement components as

$$V = \int_0^L \left\{ \frac{G_0 ab(m+3)}{(m+2)(m+1)} \left[(1+\frac{h}{2a})^{m+1} - (1-\frac{h}{2a})^{m+1} [u_{,x} + \frac{1}{2} w_{,x}^2]^2\right.\right.$$

$$
\begin{aligned}
&+\frac{b(m+2)}{4G_0 I_{yy} A(m+3)} \int_{-h/2}^{+h/2} E^2(z) I_{yy}^2 (w_{,x})^2 (\frac{z}{a}+1)^{-m} dz \\
&-\frac{G_0\alpha_0 ab(m+3)}{(m+2)(k+m+1)} \left[(1+\frac{h}{2a})^{k+m+1}-(1-\frac{h}{2a})^{k+m+1}\right][u_{,x}+\frac{1}{2}w_{,x}^2]\Delta T \Big\} dx \\
&= \int_0^L \Big\{ \frac{G_0 ab(m+3)}{(m+2)(m+1)} \left[(1+\frac{h}{2a})^{m+1}-(1-\frac{h}{2a})^{m+1}\right][u_{,x}+\frac{1}{2}w_{,x}^2]^2 \\
&+\frac{G_0 I_{yy} a(m+3)}{h(m+2)(m+1)} \left[(1+\frac{h}{2a})^{m+1}-(1-\frac{h}{2a})^{m+1}\right](w_{,x})^2 \\
&-\frac{G_0\alpha_0 ab(m+3)}{(m+2)(k+m+1)} \left[(1+\frac{h}{2a})^{k+m+1}-(1-\frac{h}{2a})^{k+m+1}\right][u_{,x}+\frac{1}{2}w_{,x}^2]\Delta T \Big\} dx
\end{aligned} \tag{13}
$$

where (,) indicates the ordinary differentiation with respect to the $x$-direction. Therefore, Eq. (13) may be shown in the following form

$$
V = \int_0^L F(u, u_{,x}, ..., w, w_{,x} ...) dx \tag{14}
$$

where $F$ is the functional of total potential energy and is given as

$$
\begin{aligned}
F &= \frac{G_0 ab(m+3)}{(m+2)(m+1)} \left[(1+\frac{h}{2a})^{m+1}-(1-\frac{h}{2a})^{m+1}\right][u_{,x}+\frac{1}{2}w_{,x}^2]^2 \\
&+\frac{G_0 I_{yy} a(m+3)}{h(m+2)(m+1)} \left[(1+\frac{h}{2a})^{m+1}-(1-\frac{h}{2a})^{m+1}\right](w_{,x})^2 \\
&-\frac{G_0\alpha_0 ab(m+3)}{(m+2)(k+m+1)} \left[(1+\frac{h}{2a})^{k+m+1}-(1-\frac{h}{2a})^{k+m+1}\right][u_{,x}+\frac{1}{2}w_{,x}^2]\Delta T
\end{aligned} \tag{15}
$$

Using the Euler equation, the minimum of Eq. (13) is obtained through the following equations

$$
\frac{\partial F}{\partial u} - \frac{d}{dx}(\frac{\partial F}{\partial u_{,x}}) = 0 \tag{16}
$$

$$
\frac{\partial F}{\partial w} - \frac{d}{dx}(\frac{\partial F}{\partial w_{,x}}) + \frac{d^2}{dx^2}(\frac{\partial F}{\partial w_{,xx}}) = 0 \tag{17}
$$

Using Eq. (15) for the definition of the functional $F$, and substituting into Eqs. (16) and (17), the equilibrium equations of a beam made of functionally graded material under thermal loading become

$$
\begin{aligned}
&\frac{2(m+3)G_0 ab}{(m+2)(m+1)} \left[(1+\frac{h}{2a})^{m+1}-(1-\frac{h}{2a})^{m+1}\right](u_{,xx}+w_{,x}w_{,xx}) \\
&\quad -\frac{(m+3)G_0\alpha_0 ab}{(m+2)(k+m+1)} \left[(1+\frac{h}{2a})^{k+m+1}-(1-\frac{h}{2a})^{k+m+1}\right]\Delta T(x)_{,x} = 0
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
&\frac{2(m+3)G_0 ab}{(m+2)(m+1)} \left[(1+\frac{h}{2a})^{m+1}-(1-\frac{h}{2a})^{m+1}\right] \\
&\quad \times[u_{,x}w_{,xx}+u_{,xx}w_{,x}+\frac{3}{2}w_{,x}^2 w_{,xx}] \\
&\quad -\frac{(m+3)G_0\alpha_0 ab}{(m+2)(k+m+1)} \left[(1+\frac{h}{2a})^{k+m+1}-(1-\frac{h}{2a})^{k+m+1}\right] \\
&\quad \times[w_{,xx}\Delta T(x)+w_{,x}\Delta T(x)_{,x}]
\end{aligned}
$$

$$-\frac{2(m+3)G_0 a I_{yy}}{(m+2)(m+1)h}\left[(1+\frac{h}{2a})^{m+1}-(1-\frac{h}{2a})^{m+1}\right]w_{,xxxx}=0 \tag{19}$$

To obtain the stability equations, the force summation method is considered. Note that whereas the equilibrium equations are nonlinear, the stability equations are linear. The components of displacement are assumed to be related to the state of stable equilibrium indicated with subscript (0), and the neighboring state (unstable equilibrium), indicated with subscript (1), as

$$u=u_0+u_1 \tag{20}$$
$$w=w_0+w_1 \tag{21}$$

Substituting relations (20) and (21) in Eqs. (18) and (19), gives

$$\begin{aligned}&\frac{2(m+3)G_0ab}{(m+2)(m+1)}\left[(1+\frac{h}{2a})^{m+1}-(1-\frac{h}{2a})^{m+1}\right][u_{0,xx}+w_{0,x}w_{0,xx}\\&\quad+u_{1,xx}+w_{0,x}w_{1,xx}+w_{1,x}w_{0,xx}+w_{1,x}w_{1,xx}]\\&\quad-\frac{(m+3)G_0\alpha_0ab}{(m+2)(k+m+1)}\left[(1+\frac{h}{2a})^{k+m+1}-(1-\frac{h}{2a})^{k+m+1}\right]\Delta T(x)_{,x}=0\end{aligned} \tag{22}$$

$$\begin{aligned}&\frac{2(m+3)G_0ab}{(m+2)(m+1)}\left[(1+\frac{h}{2a})^{m+1}-(1-\frac{h}{2a})^{m+1}\right]\\&\quad\times[u_{0,x}w_{0,xx}+u_{0,xx}w_{0,x}+\frac{3}{2}w_{0,x}^2w_{0,xx}+u_{0,x}w_{1,xx}+u_{1,x}w_{0,xx}\\&\quad+u_{1,x}w_{1,xx}+u_{0,xx}w_{1,x}+u_{1,xx}w_{0,x}+u_{1,xx}w_{1,x}+\frac{3}{2}w_{0,x}^2w_{1,xx}\\&\quad+\frac{3}{2}w_{1,x}^2w_{0,xx}+\frac{3}{2}w_{1,x}^2w_{1,xx}+3w_{0,x}w_{1,xx}w_{0,xx}+3w_{0,x}w_{1,x}w_{1,xx}]\\&\quad-\frac{(m+3)G_0\alpha_0ab}{(m+2)(k+m+1)}\left[(1+\frac{h}{2a})^{k+m+1}-(1-\frac{h}{2a})^{k+m+1}\right]\\&\quad\times[w_{0,xx}\Delta T(x)+w_{0,x}\Delta T(x)_{,x}+w_{1,xx}\Delta T(x)+w_{1,x}\Delta T(x)_{,x}]\\&\quad-\frac{2(m+3)G_0aI_{yy}}{(m+2)(m+1)h}\left[(1+\frac{h}{2a})^{m+1}-(1-\frac{h}{2a})^{m+1}\right][w_{0,xxxx}+w_{1,xxxx}]=0\end{aligned} \tag{23}$$

The stability equations of a beam made of functionally graded material under thermal loads are obtained by linearization of Eqs. (22) and (23) and by ignoring the summation of the terms as the prebuckling displacements and higher order displacement components. Applying these assumptions to Eqs. (22) and (23) results in the stability equations as

$$\frac{2(m+3)G_0ab}{(m+2)(m+1)}\left[(1+\frac{h}{2a})^{m+1}-(1-\frac{h}{2a})^{m+1}\right]u_{1,xx}=0 \tag{24}$$

$$\begin{aligned}&\frac{(m+3)G_0\alpha_0ab}{(m+2)(k+m+1)}\left[(1+\frac{h}{2a})^{k+m+1}-(1-\frac{h}{2a})^{k+m+1}\right]\\&\times[w_{1,xx}\Delta T(x)+w_{1,x}\Delta T(x)_{,x}]\\&-\frac{2(m+3)G_0aI_{yy}}{(m+2)(m+1)h}\left[(1+\frac{h}{2a})^{m+1}-(1-\frac{h}{2a})^{m+1}\right]w_{1,xxxx}=0\end{aligned} \tag{25}$$

The resulting equations may then be solved to derive the thermal buckling loads under different types of boundary conditions.

**A: Uniform temperature rise**

Consider a beam of rectangular cross section $b \times h$ and length $L$ with both ends simply supported. The boundary conditions at $x = 0$ and $x = L$ are

$$w_1 = w_{1,xx} = 0 \tag{26}$$

Therefore, the approximate solution satisfying the assumed boundary conditions is assumed as

$$w_1 = A_1 \sin \frac{n\pi x}{L} \qquad 0 \le x \le L \tag{27}$$

where $A_1$ is a constant coefficient and $n$ is number of the longitudinal buckling waves. The beam is initially at uniform temperature $T_i$ and is raised to a uniform final temperature $T_f$, such that the temperature rise is $\Delta T = T_f - T_i$. Substituting $w_1$ from Eq. (27) and the value of $\Delta T$ into Eq. (25) and make it orthogonal with respect to the approximate solution (27) according to the Galerkin method and solving for $\Delta T$, gives

$$\Delta T = \frac{h^2 n^2 \pi^2 (k+m+1)\left[(1+\frac{h}{2a})^{m+1} - (1-\frac{h}{2a})^{m+1}\right]}{6\alpha_0 L^2 (m+1)\left[(1+\frac{h}{2a})^{k+m+1} - (1-\frac{h}{2a})^{k+m+1}\right]} \tag{28}$$

The critical thermal buckling load is associated with $n = 1$, the first mode of axial buckling. The critical thermal buckling load associated with uniform temperature rise is then

$$\Delta T_{cr} = 1.645\Psi \tag{29}$$

where the parameter $\Psi$ is considered as

$$\Psi = (\frac{h}{L})^2 \, \frac{(k+m+1)\left[(1+\frac{h}{2a})^{m+1} - (1-\frac{h}{2a})^{m+1}\right]}{\alpha_0 (m+1)\left[(1+\frac{h}{2a})^{k+m+1} - (1-\frac{h}{2a})^{k+m+1}\right]} \tag{30}$$

For pure isotropic beam ($k = m = 0$), the critical thermal buckling load is

$$\Delta T_{cr} = 1.645\psi \tag{31}$$

where the parameter $\psi$ is considered as

$$\psi = \frac{1}{\alpha_0} (\frac{h}{L})^2 \tag{32}$$

Similar to the simply supported boundary condition, for the other types of five boundary conditions, the critical thermal buckling loads for the functionally graded and pure isotropic beam are given, respectively

$$\Delta T_{cr} = \lambda_1 \Psi \tag{33}$$

$$\Delta T_{cr} = \lambda_1 \psi \tag{34}$$

The coefficient $\lambda_1$ is known and is given in Table (1).

Table (1). The values of $\lambda_1$ for uniform temperature rise and six types of boundary conditions.

| Boundary conditions | Approximate solution | $\lambda_1$ |
|---|---|---|
| Simply supported beam (S-S) $w_1 = w_{1,xx} = 0$ at $x = 0$ and $x = L$ | $w_1 = A_1 \sin n\pi x/L$ | 1.645 |
| Clamped-clamped beam (C-C) $w_1 = w_{1,x} = 0$ at $x = 0$ and $x = L$ | $w_1 = A_1 \sin^2 n\pi x/L$ | 6.58 |
| Roller-roller beam (R-R) $w_{1,x} = w_{1,xxx} = 0$ at $x = 0$ and $x = L$ | $w_1 = A_1 \cos n\pi x/L$ | 1.645 |
| Simply supported-clamped beam (S-C) $w_1 = w_{1,x} = 0$ at $x = 0$ $w_1 = w_{1,xx} = 0$ at $x = L$ | $w_1 = -n\pi x^3/L^3 + 3n\pi x^2/L^2$ $-2n\pi x/L + \sin 2n\pi x/L$ | 1.1 |
| clamped-roller beam (C-R) $w_1 = w_{1,x} = 0$ at $x = 0$ $w_{1,x} = w_{1,xxx} = 0$ at $x = L$ | $w_1 = 4n^3\pi^3 x^3/3L^3 - 2n^3\pi^3 x^2/L^2$ $-2n\pi x/L + \sin 2n\pi x/L$ | 1.66 |
| Simply supported-roller beam (S-R) $w_1 = w_{1,xx} = 0$ at $x = 0$ $w_{1,x} = w_{1,xxx} = 0$ at $x = L$ | $w_1 = 4n^3\pi^3 x^3/3L^3 - 4n^3\pi^3 x/L$ $-2n\pi x/L + \sin 2n\pi x/L$ | 0.416 |



**B: Axial temperature difference**

Consider a linear axial temperature distribution along the length of the beam as

$$T(x) = \frac{T_2 - T_1}{L}\, x + T_1 \tag{35}$$

where $T_1$ and $T_2$ are the temperatures at ends $x = 0$ and $x = L$, respectively. Both ends of the beam are simply supported. Substituting Eqs. (35) and (27) into Eq. (25) and make it orthogonal with respect to the approximate solution (27) according to the Galerkin method and solving for $\Delta T$, gives

Table (2) The values of $\lambda_2$ for axial temperature difference and six types of boundary conditions

| Boundary conditions | Approximate solution | $\lambda_2$ |
|---|---|---|
| Simply supported beam (S-S) $w_1 = w_{1,xx} = 0$ at $x = 0$ and $x = L$ | $w_1 = A_1 \sin n\pi x/L$ | 3.29 |
| Clamped-clamped beam (C-C) $w_1 = w_{1,x} = 0$ at $x = 0$ and $x = L$ | $w_1 = A_1 \sin^2 n\pi x/L$ | 13.16 |
| Roller-roller beam (R-R) $w_{1,x} = w_{1,xxx} = 0$ at $x = 0$ and $x = L$ | $w_1 = A_1 \cos n\pi x/L$ | 3.29 |
| Simply supported-clamped beam (S-C) $w_1 = w_{1,x} = 0$ at $x = 0$ $w_1 = w_{1,xx} = 0$ at $x = L$ | $w_1 = -n\pi x^3/L^3 + 3n\pi x^2/L^2$ $-2n\pi x/L + \sin 2n\pi x/L$ | 2.2 |
| clamped-roller beam (C-R) $w_1 = w_{1,x} = 0$ at $x = 0$ $w_{1,x} = w_{1,xxx} = 0$ at $x = L$ | $w_1 = 4n^3\pi^3 x^3/3L^3 - 2n^3\pi^3 x^2/L^2$ $-2n\pi x/L + \sin 2n\pi x/L$ | 3.32 |
| Simply supported-roller beam (S-R) $w_1 = w_{1,xx} = 0$ at $x = 0$ $w_{1,x} = w_{1,xxx} = 0$ at $x = L$ | $w_1 = 4n^3\pi^3 x^3/3L^3 - 4n^3\pi^3 x/L$ $-2n\pi x/L + \sin 2n\pi x/L$ | 0.832 |

$$\Delta T_{cr} = 3.29\Psi \tag{36}$$

For pure isotropic beam ($k = m = 0$), the critical thermal buckling load is

$$\Delta T_{cr} = 3.29\psi \tag{37}$$

where the parameters $\Psi$ and $\psi$ are defined by the same Eqs. (30) and (32). Similar to the simply supported boundary condition, for all six types of boundary conditions, the critical thermal buckling loads for the functionally graded and pure isotropic beam are given, respectively

$$\Delta T_{cr} = \lambda_2\Psi \tag{38}$$

$$\Delta T_{cr} = \lambda_2\psi \tag{39}$$

The coefficient $\lambda_2$ is known and is given in Table (2). It is interesting to note that the buckling temperatures associated with the axial temperature difference are, in all cases, twice those of uniform temperature rise.

## 3 Results and discussions

Consider a ceramic-metal functionally graded beam. Assume that the top surface of the beam is ceramic, and the bottom surface of the beam is metal. The combination of materials consist of steel and alumina. Young's modulus and thermal expansion coefficient for steel are: $E_m = 200\ GPa$, $\alpha_m = 11.7 \times 10^{-6}\ 1/oC$, and for alumina are $E_c = 380\ GPa$, $\alpha = 7.4 \times 10^{-6}\ 1/oC$, respectively. Assume that the characteristic length is $a = h$, where $h$ is the height of the beam cross section. In Eqs. (1) and (2) the coefficients $\alpha_0$, $G_0$, $k$, and $m$ are unknowns and are obtained from the following conditions

$$\alpha\ (\frac{h}{2}) = \alpha_c \qquad \alpha\ (-\frac{h}{2}) = \alpha_m \tag{40}$$

$$G\ (\frac{h}{2}) = G_c \qquad G\ (-\frac{h}{2}) = G_m \tag{41}$$

Therefore, the following expressions are found as

$$(\frac{3}{2})^k \alpha_0 = 7.4 \times 10^{-6} \qquad (\frac{1}{2})^k \alpha_0 = 11.7 \times 10^{-6} \tag{42}$$

$$(\frac{3}{2})^m G_0 = 146.154 \qquad (\frac{1}{2})^m G_0 = 76.923 \tag{43}$$

The above is a non-linear system of equations, which upon solution gives $k = -0.417$, $m = 0.584$, $\alpha_0 = 8.768 \times 10^{-6}\ 1/oC$, and $G_0 = 115.354\ GPa.$ Note that the value of Poisson's ratio is $\nu = 0.387$. Substituting these values into relations of $\Delta T_{cr}$ for all cases provide the critical thermal buckling loads of the ceramic-metal functionally graded beam.

Figures (1) and (2) show the critical normalized buckling temperature difference of a pure isotropic beam under uniform temperature rise and axial temperature difference, respectively. It is seen that the critical thermal buckling load increases with the increase of $(h/L)$. The clamped-clamped boundary condition has the highest buckling temperature, and the simply supported-roller boundary condition has the lowest buckling temperature.

Figures (3) and (4) show the variation of the critical normalized buckling temperature difference versus the variation of the FGM parameter $k$ and $m$, and are plotted for simply supported beam. It is seen that the critical thermal buckling load decreases with the increase of $k$ and $m$. Penetration and influence of $m$ is more effective than $k$. These figures are plotted for $h/L = 0.05$.

Figures (5) and (6) show the variation of the critical normalized buckling temperature difference versus the variation of $h/L$ for different values of $k$ and $m$ and for a pure isotropic and functionally graded beam. These figures are plotted for simply supported boundary condition. It is seen that the critical buckling temperature difference for the functionally graded beam is lower than the corresponding value for the pure isotropic beam. This is due to the inherent different coefficients of thermal expansion of the constituent materials, which result into thermal bending upon temperature change.

## 4 Conclusion

As conclusion, we may refer to the following points:

1) The thermal buckling loads are directly proportional to $(h/L)$ and inversely proportional to the coefficient of thermal expansion of the beam on the neutral plane $\alpha_0$.
2) The critical thermal buckling load for the functionally graded beam is generally lower than

the corresponding value for the pure isotropic beam.
3) The highest critical thermal buckling load belongs to the clamped-clamped boundary condition and the lowest critical thermal buckling load belongs to the simply supported-roller boundary condition.
4) The buckling temperature associated with the linear axial temperature difference are in all cases twice those of uniform temperature rise.
5) The critical thermal buckling load of functionally graded beam is independent of width of the beam.
6) The critical thermal buckling load is decreased by increasing the power indexes $m$ and $k$ of shear modulus and thermal expansion coefficient functions, respectively.
7) The critical thermal buckling load is directly proportional to the number of longitudinal buckling waves $n$.
8) The values of critical thermal buckling loads for simply supported and roller-roller boundary conditions are identical.
9) The values of critical thermal buckling loads for functionally graded beam and pure isotropic beam are identical for special values of $k$ and $m$.
10) Penetration and influence of shear modulus $G(z)$ is more effective than the thermal expansion coefficient $\alpha(z)$ on the critical thermal buckling load.

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**Nomenclature**

$A$: cross sectional area of beam
$a$: characteristic length
$b$: width of beam
$E(z)$: elastic modulus varying continuous function of beam made of FGM
$F$: functional of total potential energy
$G_0$: shear modulus on the neutral surface of beam made of FGM
$G(z)$: shear modulus varying continuous function of beam made of FGM
$h$: thickness of beam made of FGM
$I_{yy}$: second moment of inertia of cross sectional area of beam made of FGM
$k$: power index in thermal expansion coefficient function of beam made of FGM
$K_y$: curvature of beam made of FGM in $x-z$ plane
$L$: length of beam made of FGM
$M(x)$: moment function in $x$-direction
$m$: power index in shear modulus function of beam made of FGM
$n$: number of the longitudinal buckling waves
$T_1$: temperature at $(x=0)$
$T_2$: temperature at $(x=L)$,
$T(x)$: temperature distribution function in $x$-direction
$U_b$: bending strain energy
$U_m$: membrane strain energy
$U_T$: thermal strain energy
$u_0$: prebuckling axial displacement

$u_1$: axial displacement neighboring state
$u$: axial displacement
$V$: total strain energy
$w_0$: prebuckling lateral displacement
$w_1$: lateral displacement neighboring state
$w$: lateral displacement
$\alpha_0$:thermal expansion coefficient on the neutral surface of beam made of FGM
$\epsilon_T$: thermal expansion coefficient varying continuous function of beam made of FGM = $\alpha(z)$
axial thermal strain
$\epsilon_x$: axial normal strain
$\nu$: Poisson's ratio
$\sigma_x$: axial normal stress
$\Delta T$: temperature difference
$\Delta T_{cr}$: critical temperature difference (proportional to critical thermal buckling load)

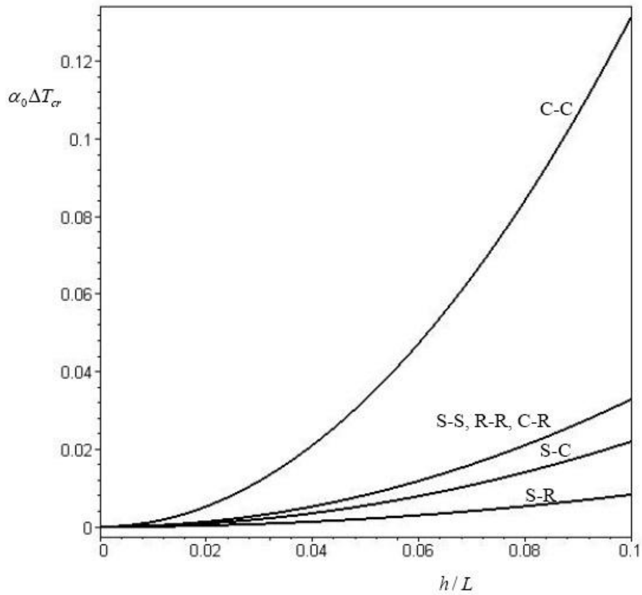


**Figure 1** Normalized buckling temperature difference of a pure isotropic beam under axial temperature difference.

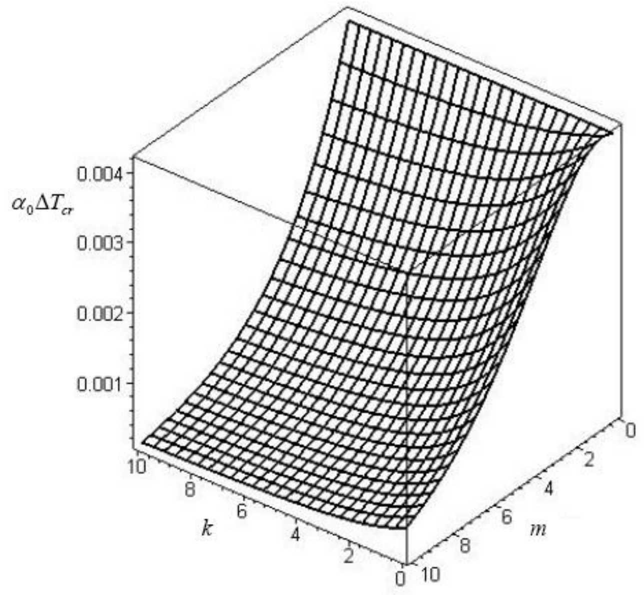


**Figure 2** Variation of the thermal buckling temperature versus the FGM parameters $k$ and $m$ ($h/L = 0.05$, uniform temperature rise).

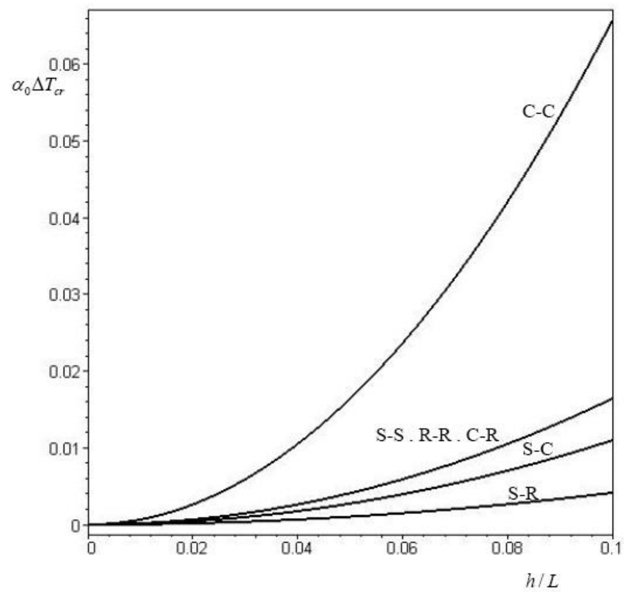


**Figure 3** Normalized buckling temperature difference of a pure isotropic beam under uniform temperature rise.

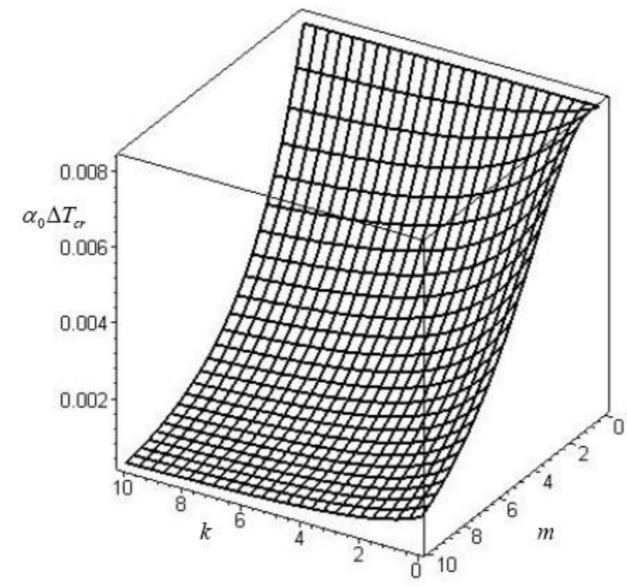


**Figure 4** Variation of the thermal buckling temperature versus the FGM parameters $k$ and $m$ ($h/L = 0.05$, axial temperature difference).

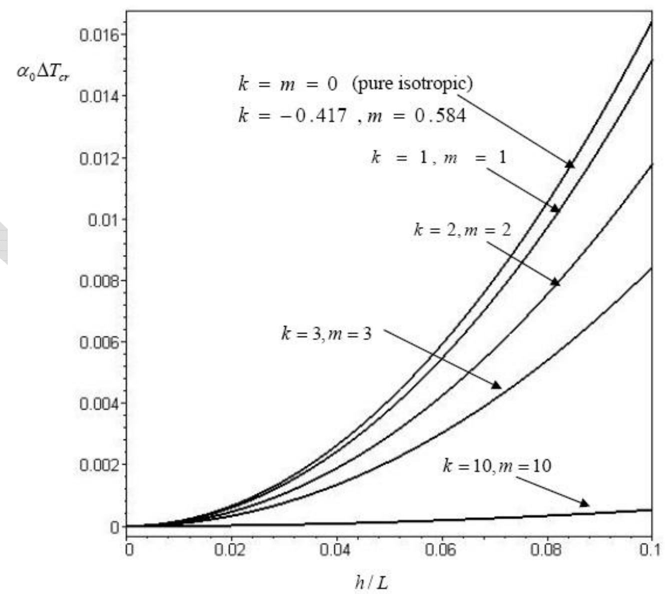


**Figure 5** Variation of the buckling temperature vs $h/L$ for different values of $k$ and $m$ (uniform temperature rise).

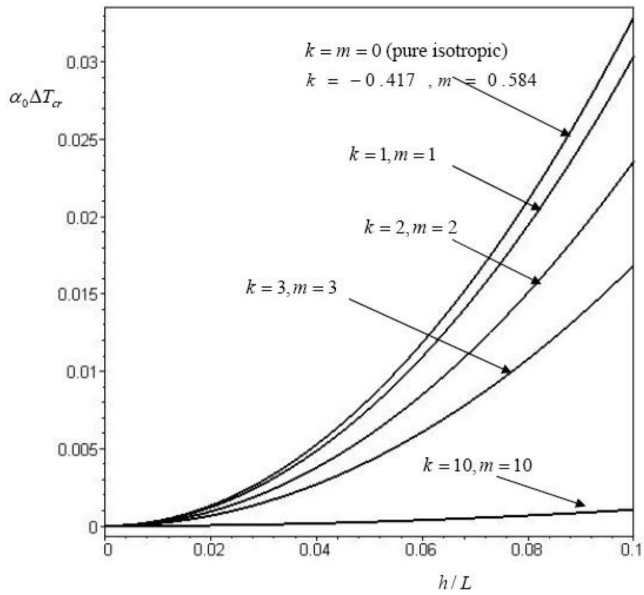


**Figure 6** Variation of the buckling temperature vs $h/L$ for different values of $k$ and $m$ (axial temperature difference).

## چکیده

در این مقاله کمانش حرارتی تیرهای ساخته شده از مواد تابعی در نظر گرفته شده است. معادلات بر اساس تئوری یک بعدی الاستیسیته بدست آمده‌اند. خصوصیت مواد به صورت پیوسته در راستای ضخامت تیر متغیر است. مدل تانیگاوا برای تغییرات ضریب پواسون، مدول برشی و ضریب انبساط حرارتی در نظر گرفته شده است. معادلات تعادل و پایداری تیر تابعی تحت بارگذاری حرارتی با استفاده از روش حساب تغییرات و جمع نیروها بدست آمده‌اند. یک تیر با شش شرط مرزی مختلف در نظر گرفته شده و حلهای بسته برای بار کمانش بحرانی بدون بعد وابسته به افزایش یکنواخت درجه حرارت و تغییرات درجه حرارت محوری بدست آمده است. نتایج برای بدست آوردن فرمول کمانش تیرهای ساخته شده از مواد ایزوتروپیک ساده‌سازی شده‌اند.