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Some New Analytical Techniques for Duffing Oscillator with Very Strong Nonlinearity

A. Farshidianfar [1]
Assosiate Professor

N.Nickmehr [2]
Gradute

The current paper focuses on some analytical techniques to solve the non-linear Duffing oscillator with large nonlinearity. Four different methods have been applied for solution of the equation of motion; the variational iteration method, He's parameter expanding method, parameterized perturbation method, and the homotopy perturbation method.
The results reveal that approximation obtained by these approaches are valid uniformly even for very large parameters and are more accurate than straightforward expansion solution.
The methods, which are proved to be mathematically powerful tools for solving the nonlinear oscillators, can be easily extended to any nonlinear equation, and the present paper can be used as paradigms for many other applications in searching for periodic solutions, limit cycles or other approximate solutions for real-life physics and engineering problems.

Keyword: Duffing oscillator; strong nonlinearity; parameterized perturbation method; parameter expanding method; variational iteration method; homotopy perturbation;

1 Introduction

In reality, virtually every process is a nonlinear system and described by nonlinear equations. After the appearance of the computers, it is not difficult to find the solution of the linear problems. However it is still difficult to solve nonlinear systems analytically.

Gaëtan Kerschen et al. [1], studied the typical sources of nonlinearity in their review paper and categorized them as follows:

1- Geometric nonlinearity results when a structure undergoes large displacements.

[1] Corresponding author, Department of Mechanical Engineering Ferdowsi University of Mashhad, Email : Farshid@un.ac.ir
[2] Department of Mechanical Engineering Ferdowsi University of Mashhad

2- Inertia nonlinearity derives from nonlinear terms containing velocities and/or accelerations in the equations of motion, and takes its source in the kinetic energy of the system.
3- A nonlinear material behavior may be observed when the constitutive law relating stresses and strains is nonlinear.
4- Damping sources other than linear viscous damping, introduce nonlinear effects and include hysteresis, drag, and coulomb friction.
5- Nonlinearity may also results due to boundary conditions or certain external nonlinear body forces.

Nayfeh and Mook in1979 [2], Strogatz in 1994 [3], Verhulst in 1999 [4] and Rand in 2003 [5] studied the nonlinear oscillations.

Nonlinear problems can be solved numerically and analytically, but obtaining analytical solution for nonlinear systems is very important due to limitations of numerical methods.

It is well-known that the common weakness of equivalent linearization approaches [6,7], restricts them to solve problem with weak nonlinearities and within a narrow range of parametric variations, so many efforts have been made to develop the method for studying the nonlinear systems. The concept of mode shapes has been proposed by Rosenberg [8,9]; Then Rand, Shaw and Pierre, and Vakakis et al, have modified this method [10-13].

With the rapid development of nonlinear science, there appears an ever-increasing interest of scientists and engineers in the analytical asymptotic techniques for nonlinear problems. Though it is very easy for us now to find the solutions of linear systems by means of computer, it is, however, still very difficult to solve nonlinear problems either numerically or theoretically. This is possibly due to the fact that the various discredited methods or numerical simulations apply iteration techniques to find their numerical solutions of nonlinear problems, and nearly all iterative methods are sensitive to initial solutions, so it is very difficult to obtain converged results in cases of strong nonlinearity. In addition, the most important information, such as the natural circular frequency of a nonlinear oscillation depends on the initial conditions (i.e. amplitude of oscillation) will be lost during the procedure of numerical simulation.

Perturbation methods provide the most versatile tools available in nonlinear analysis of engineering problems and they are constantly being modified and applied to ever more complex problems. But perturbation methods have their own restrictions as well as other nonlinear techniques (Nayfeh and Mook in1979 [2], Nayfeh in 1981 [14], O'Malley in 1991 [15] and Kevorkian an Cole in 1996 [16]). Almost all perturbation methods are based on such an assumption that a small parameter must exist in an equation. This so-called small parameter assumption greatly restricts applications of perturbation techniques, as is well known, an overwhelming majority of nonlinear problems, especially those having strong nonlinearity, have no small parameters at all.

Regarding the above descriptions, some new developed methods valid for the large parameter domain, should be introduced to solve nonlinear problems.

Duffing equation is widely used in the papers for verifying the effectiveness of various techniques. The main target of the present investigation is to solve Duffing equation with different developed analytical methods that are proposed by Ji-Huan He [17, 18].

2 Solving Duffing equation by four different methods

2.1 Variational iteration method

A new kind of analytical technique for a nonlinear problem called the variational iteration method is described and used to give approximate solutions for some well-known nonlinear problems by Ji-Huan He in1999 [17]. In this method we start with an initial estimate that is gained from linearization of the problem and then a more highly precise approximation can be obtained.

The general nonlinear system has the following form:

$$Lu + N\,u = g(x) \tag{1}$$

where L is a linear operator and N is a nonlinear operator.

Considering $u_0(x)$ is the solution of Lu=0 [19], the following relation can be expressed to correct the value of some especial points, for example at x=1:

$$u_{cor}(1) = u_0(1) + \int_0^1 \lambda (Lu_0 + Nu_0 - g)\,dx \tag{2}$$

where λ is a general Lagrange multiplier [19], which can be determined via the variational theory. The integral term is the correction expression. He has developed the above method by using an iteration procedure as follows [20]:

$$u_{n+1}(x_0) = u_n(x_0) + \int_0^{x_0} \lambda (L\,u_n + N\,\tilde{u}_n - g)\,dx \tag{3}$$

Assume $u_0(x)$ as the first estimate with possible unknowns, and $\tilde{u}_n$ as a restricted variation [21], i.e. $\delta\,\tilde{u}_n = 0$. For arbitrary value of x_0, we can rewrite equation (3) as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^{x} \lambda (Lu_n(\zeta) + N\tilde{u}_n(\zeta) - g(\zeta))\,d\zeta \tag{4}$$

Equation (4) is called a correction functional. The modified method, or variational iteration method has been shown to solve effectively, easily, and accurately a large class of non-linear problems with approximations converging rapidly to accurate solutions [20].

Now, Duffing equation with fifth nonlinearity may be solved by the variational iteration method. Consider the following equation with the given initial conditions:

$$\frac{d^2u}{dt^2}+u+\varepsilon u^5 = 0, \quad u(0) = A, \ \ u'(0) = 0 \tag{5}$$

Its correction functional can be given as follows:

$$u_{n+1}(t)=u_n(t)+\int_0^t \lambda\left(\frac{d^2u_n(\tau)}{d\tau^2}+u_n(\tau)+\varepsilon\tilde{u}_n^{\,5}(\tau)\right)d\tau \tag{6}$$

where $\tilde{u}_n$ is considered as a limited variation. Making the above correction functional stationary, and noticing that $\delta u(0) = 0$:

$$\begin{aligned}\delta u_{n+1}(t)&=\delta u_n(t)+\delta\int_0^t \lambda\left(\frac{d^2u_n(\tau)}{d\tau^2}+u_n(\tau)+\varepsilon\tilde{u}_n^{\,5}(\tau)\right)d\tau\\ &= \delta u_n(t)+\lambda(\tau)\delta y'_n(\tau)|_{\tau=t}-\lambda'(\tau)\delta y_n(\tau)|_{\tau=t}\\ &+\int_0^t\left(\frac{d^2u_n(\tau)}{d\tau^2}+u_n(\tau)\right)\delta y_n|d\tau=0\end{aligned} \tag{7}$$

yields the following stationary conditions:

$$\begin{aligned}&\delta y_n : \lambda''(\tau)+\lambda(\tau)=0,\\ &\delta y'_n : \lambda(\tau)|_{\tau=t}=0,\\ &\delta y_n : 1-\lambda'(\tau)|_{\tau=t}=0\end{aligned} \tag{8}$$

Thus the multiplier can be determined as $\lambda=\sin(\tau-t)$, and the variational iteration formula can be obtained:

$$u_{n+1}(t)=u_n(t)+\int_0^t \sin(\tau-t)\times\left\{\frac{d^2u_n(\tau)}{d\tau^2}+u_n(\tau)+\varepsilon u_n^{\,5}(\tau)\right\}d\tau \tag{9}$$

Assuming that its initial trial has the form:

$$u_0(t)=A\cos(\alpha t) \tag{10}$$

where α (ε) is a non-zero unknown function of ε with α (0) = 1. Substituting equation (10) in equation (5) yields the following residual:

$$R_0(t)=(-\alpha^2+1+\frac{5}{8}\varepsilon A^4)A\cos\alpha t+\frac{1}{16}\varepsilon A^5(\cos 5\alpha t+5\cos 3\alpha t) \tag{11}$$

Using variational iteration formula (9), result in:

$$\begin{aligned} u_1(t) &= A\cos\alpha t+\int_0^t \sin(\tau-t)R_0(\tau)\,d\tau= \\ &= A\cos\alpha t+(-\alpha^2+1+\frac{5}{8}\varepsilon A^4)\times\frac{A}{a^2-1}(\cos\alpha t-\cos t) \\ &+\frac{5\varepsilon A^5}{16(9a^2-1)}(\cos 3\alpha t-\cos t) \\ &+\frac{\varepsilon A^5}{16(25a^2-1)}(\cos 5\alpha t-\cos t) \end{aligned} \tag{12}$$

Due to no appearance of secular terms in the next iteration, resonance must be avoided. Thus the coefficient of *cos t* is considered to be zero:

$$-\frac{5\varepsilon A^5}{16(9\alpha^2-1)}-\frac{\varepsilon A^5}{16(25\alpha^2-1)}-(-\alpha^2+1+\frac{5}{8}\varepsilon A^4)\times\frac{A}{\alpha^2-1}=0 \tag{13}$$

After solving equation (13) and determining α, the first order approximation can be written down as follows:

$$\begin{aligned} &\alpha=\sqrt{1+\frac{5}{8}\varepsilon A^4+O(\varepsilon^2 A^8)} \\ &u_1(t)=\frac{5\varepsilon A^5}{8(a^2-1)}\cos\alpha t+\frac{5\varepsilon A^5}{16(9a^2-1)}\cos 3\alpha t+\frac{\varepsilon A^5}{16(25a^2-1)}\cos 5\alpha t \end{aligned} \tag{14}$$

Hence the approximate period that is true for very large parameter equals to:

$$T=\frac{2\pi}{\sqrt{1+\frac{5}{8}\varepsilon A^4}} \tag{15}$$

and the period which found from perturbation method that is valid only for small parameter ε [14], reads:

$$T = 2\pi(1 - 5\varepsilon A^4 / 16) \tag{16}$$

2.2 *He's Parameter expanding method*

Parameter-expanding methods including the modified Lindstedt-Poincare method and bookkeeping parameter method can successfully deal with such special cases, however the classical methods fail. The methods need not have a time transformation like Lindstedt-Poincare method; the basic character of the method is to expand the solution and some parameters in the equation.

A general nonlinear oscillator can be expressed by the following equation:

$$\begin{aligned} &u'' + au + bu^3 + cu^{1/3} = 0 \\ &u(0) = A \qquad , \; u'(0) = 0 \end{aligned} \tag{17}$$

If $a+bA^2+cA^{-2/3}>0$, the above equation has periodic solution [18]. In case $a<=0$, one cannot use the traditional perturbation methods even when the parameters b and c are small.

The solution is expanded into a series of an artificial parameter, p, in the He's parameter expanding method (PEM) [18, 22]:

$$u = u_0 + pu_1 + p^2 u_2 + \cdots \tag{18}$$

where p is a bookkeeping parameter.

Also, the coefficient a, b and c can be expanded into a series in p in a similar way [23]:

$$\begin{aligned} &a = \omega^2 + p\,\omega_1 + p^2\,\omega_2 + \cdots \\ &b = p\,b_1 + p^2\,b_2 + \cdots \\ &c = p\,c_1 + p^2\,c_2 + \cdots \end{aligned} \tag{19}$$

Substituting equations (18) and (19) into equation (17), collecting terms of the same power of p, gives:

$$\begin{aligned} &p^0 = u_0'' + \omega^2 u_0 = 0 \\ &p^1 = u_1'' + \omega^2 u_1 + \omega_1 u_0 + b_1 u^3{}_0 + c_1 u_0{}^{1/3} = 0 \end{aligned} \tag{20}$$

According to initial conditions of equation (17), the solution of the first equation of expression (20) is $u_0(t)=A\cos(\omega t)$.
Substituting the results into the second equation of expression (20), yields:

$$u_1''+\omega^2 u_1+\omega_1 A\cos\omega t+\frac{3}{4}b_1 A^3\cos\omega t+\frac{1}{4}b_1 A^3\cos 3\omega t+c_1 A^{1/3} a_1 (\cos\omega t)^{1/3}=0 \quad (21)$$

Then the term $(\cos\omega t)^{1/3}$ is expanded into a Fourier series:

$$(\cos\omega t)^{1/3}=\sum_{n=0}^{\infty} a_{2n+1}\cos(2n+1)\omega t \quad (22)$$

in which $a_{2n+1}=\dfrac{3\Gamma(\frac{7}{3})}{2^{4/3}\,\Gamma(n+\frac{5}{3})\Gamma(\frac{2}{3}-n)}$ with a_1 =1.15959526696 and the interval of t in equation (22) is [-π/ω, π/ω]. Thus the first several terms are

$$(\cos\omega t)^{1/3}=a_1\left(\cos\omega t-\frac{\cos 3\omega t}{5}+\frac{\cos 5\omega t}{10}-\frac{7\cos 7\omega t}{110}+\cdots\right) \quad (23)$$

By substituting equation (23) into equation (21), the following relation will be obtained:

$$\begin{aligned}&u_1''+\omega^2 u_1+\omega_1 A\cos\omega t+\frac{3}{4}b_1 A^3\cos\omega t+\\&\frac{1}{4}b_1 A^3\cos 3\omega t+c_1 A^{1/3} a_1\left(\cos\omega t-\frac{\cos 3\omega t}{5}+\cdots\right)=0\end{aligned} \quad (24)$$

No secular terms in u_1 requires that:

$$\omega_1 A+\frac{3}{4}b_1 A^3+c_1 A^{1/3} a_1=0 \quad (25)$$

If the first order approximation is considered, then p is set to be 1 in equation (19) and results:

$$\begin{aligned}a&=\omega^2+\omega_1\\b&=b_1\\c&=c_1\end{aligned} \quad (26)$$

Solving equations (25) and (26) yields:

$$\omega=\sqrt{\frac{3}{4}b\,A^2+1.15959526696\,c\,A^{-2/3}+a} \tag{27}$$

Now the Duffing equation with third nonlinearity is considered and is solved by this method. If a = 1, b = ε and c = 0, according to equation (17), the Duffing oscillator can be obtained as follows:

$$\begin{aligned}&\frac{d^2u}{dt^2}+u+\varepsilon u^3=0,\\&u(0)=A,\ \ u'(0)=0\end{aligned} \tag{28}$$

Then regarding to equation (27), the frequency of the above nonlinear Duffing oscillator can be calculated:

$$\omega=\sqrt{\frac{3}{4}\varepsilon\,A^2+1} \tag{29}$$

Therefore zero-order approximate solution can be determined as follows:

$$u=A\cos(\sqrt{\frac{3}{4}\ A^2+1}\,t) \tag{30}$$

2.3 Parametrized *Perturbation Method*

To describe this method one begins with Duffing equation initially:

$$\begin{aligned}&\frac{d^2u}{dt^2}+u+\alpha u^5=0,\\&u(0)=A,\ \ u'(0)=0\end{aligned} \tag{31}$$

in which $0\le\alpha<\infty$ and need not to be small.

Consider:

$$u=\varepsilon v \tag{32}$$

Substitute the above relation into equation (31) and obtain:

$$v'' + 1\cdot v + \alpha\varepsilon^4 v^5 = 0\ ,\ \ v(0) = A/\varepsilon\ \ ,\ \ v'(0) = 0 \tag{33}$$

By using the parameter expanding method (modified Lindstedt - Poincare method [24]), one can assume that the solution of equation (33) and the constant 1 can be expanded in the forms[2]:

$$\begin{aligned} v &= v_0 + \varepsilon^4 v_1 + \varepsilon^8 v_2 + \cdots \\ 1 &= \omega^2 + \varepsilon^4 \omega_1 + \varepsilon^8 \omega_2 + \cdots \end{aligned} \tag{34}$$

Substituting equation (34) into equation (33) and equating coefficients of like powers of ε, yields the following equations:

$$v_0'' + \omega^2 v_0 = 0\ ,\ \ v(0) = A/\varepsilon\ \ ,\ \ v'(0) = 0 \tag{35}$$

$$v_1'' + \omega^2 v_1 + \omega_1 v_0 + \alpha v^5{}_0 = 0\ ,\ \ v_1(0) = 0\ \ ,\ \ v_1'(0) = 0 \tag{36}$$

The solution of equation (35) is:

$$v_0(t) = A/\varepsilon \cos(\omega t) \tag{37}$$

By substituting $v_0(t)$ into equation (36):

$$v_1'' + \omega^2 v_1 + \left(\frac{5\alpha A^4}{8\varepsilon^4} + \omega_1\right)\frac{A}{\varepsilon}\cos\omega t + \frac{5\alpha A^5}{16\varepsilon^5}\cos 3\omega t + \frac{\alpha A^5}{16\varepsilon^5}\cos 5\omega t = 0 \tag{38}$$

Avoiding the presence of a secular term needs:

$$\omega_1 = -\frac{5\alpha A^4}{8\varepsilon^4} \tag{39}$$

Therefore, the response of equation (38) is obtained as follows:

$$u_1 = -\frac{\alpha A^5}{128\varepsilon^5\omega^2}(\cos\omega t - \cos 3\omega t) - \frac{\alpha A^5}{384\varepsilon^5\omega^2}(\cos\omega t - \cos 5\omega t) \tag{40}$$

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[2] Note: If we suppose that

$$v = v_0 + \varepsilon\, v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \varepsilon^4 v_4 + \cdots \quad and \qquad 1 = \omega^2 + \varepsilon\, \omega_1 + \varepsilon^2 \omega_2 + \varepsilon^3 \omega_3 + \varepsilon^4 \omega_4 + \cdots$$

Then it is easy to find that $v_1 = v_2 = v_3 = 0$ and $\omega_1 = \omega_2 = \omega_3 = 0$, so that the secular terms will not occur.

If, for example, its first-order approximation is sufficient, then:

$$u=\varepsilon v=\varepsilon(v_0+\varepsilon^4 v_1)=A\cos\omega t-\frac{\alpha A^5}{128\omega^2}(\cos\omega t-\cos 3\omega t)-\frac{\alpha A^5}{384\omega^2}(\cos\omega t-\cos 5\omega t) \qquad (41)$$

Thus, the frequency of the oscillator can be calculated in the form:

$$\omega=\sqrt{1-\varepsilon^4\omega_1}=\sqrt{1+\frac{5}{8}\alpha A^4} \qquad (42)$$

the above relation is correct for all $\alpha > 0$.

2.4 *Homotopy Perturbation Method*

Homotopy perturbation method is a relatively new method, it is still evolving. Like other methods, it has theoretical and application limitations. The homotopy perturbation technique does not depend upon a small parameter in the equation. By the homotopy technique in topology, a homotopy is constructed with an imbedding parameter $p\in$ [0, 1], which is considered as a ‘‘small parameter’’.

In this section, homotopy perturbation method will be used for solving the following Duffing equation:

$$\frac{d^2u}{dt^2}+u+\varepsilon u^3=0, \qquad u(0)=A,\ \ u'(0)=0 \qquad (43)$$

Now the homotopy is constructed in the form:

$$u''+\omega^2 u+p(\varepsilon u^3+(1-\omega^2)u)=0, \qquad p\in[0,1] \qquad (44)$$

If p=0, the linearized equation will be $u''+\omega^2 u=0$ and when p=1 the above equation turns out to be original one. Suppose that the periodic solution for equation (44) can be expanded as a power series in p:

$$u=u_0+pu_1+p^2u_2+\cdots \qquad (45)$$

Substituting equation (45) into equation (44) and equating the terms with the identical powers of p:

$$\begin{aligned} &u_0'' + \omega^2 u_0 = 0 \\ &u_0(0) = A, \quad u_0'(0) = 0 \end{aligned} \tag{46}$$

$$\begin{aligned} &u_1'' + \omega^2 u_1 + \varepsilon u_0{}^3 + (1-\omega^2) u_0 = 0 \\ &u_1(0) = 0 \ , \ u_1'(0) = 0 \end{aligned} \tag{47}$$

Solution of equation (46) is $u_0(t) = A\cos(\omega t)$ and substituting this into equation (47), results:

$$u_1'' + \omega^2 u_1 + (1 + 3/4\,\varepsilon A^2 - \omega^2) A\cos\omega t + 1/4\,\varepsilon A^3 \cos 3\omega = 0 \tag{48}$$

No secular terms in u_1 requires that:

$$\omega = \sqrt{\frac{3}{4}\varepsilon A^2 + 1} \tag{49}$$

3 Results and Discussion

The exact solution for equation (5) can be readily obtained as follows [18]:

$$T_{ex} = \frac{4}{\sqrt{1+\frac{1}{3}\varepsilon A^4}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 + k\cos^2 x + k\cos^4 x}} \tag{50}$$

with $k = \frac{1}{3}\varepsilon A^4 / (1 + \frac{1}{3}\varepsilon A^4)$.

Thus, in the case $\varepsilon \to \infty$, the accuracy of the calculated period based on the *variational iteration method* and the *parameterized perturbation method*, is derived as follows:

$$\begin{aligned} \lim_{\varepsilon\to\infty} \frac{T_{exa}}{T} &= \frac{2\sqrt{\frac{15}{8}}}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1+\cos^2 x + \cos^4 x}} \\ &= \frac{2\sqrt{\frac{15}{8}}}{\pi} \times 1.14811 = 1.0008 \end{aligned} \tag{51}$$

Therefore, for any value of ε, it can be easily proved that $0 \le \left|(T_{ex} - T)\right| \le 0.08\%$, thus the approximate solution determined by these methods, is uniformly true for any value of ε, as mentioned before.

In Figure 1 the exact solution and the proposed responses that are obtained from *variational iteration method* and *parameterized perturbation method* have been plotted for A=1 and ε=1, and compared which each other. It has been showed very good agreement.

Now, to determine the exact period of equation (28), the following relation can be used [18]:

$$T_{ex} = \frac{4}{\sqrt{1+\varepsilon A^2}} \int_0^{\pi/2} \frac{dx}{\sqrt{1-k\sin^2 x}} \qquad (52)$$

where $k = \frac{1}{2}\varepsilon A^2 \,/\,(1+\varepsilon A^2)$

Therefore, the accuracy of the periods that was calculated by using *He's parameter expanding method* and *homotopy perturbation method* is:

$$\lim_{\varepsilon \to \infty} \frac{T_{exa}}{T} = \frac{2\sqrt{\frac{3}{4}}}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1-0.5\sin^2 x}} = 0.9294 \qquad (53)$$

As the above relation shows, the maximal relative error is less than 7.06%, thus the approximate solution for Duffing equation with third nonlinearity based on these methods are true for any value of *ε*.

Figure 2 shows the results obtained from the exact solution and the above methods. It is clear that there is no considerable difference between these methods, which approves the new applied methods. Due to the very high accuracy of the first-order approximate solution, one can stop the procedure before the second iteration step.

Both variational iteration method and parameterized perturbation method have been shown to solve effectively, easily, and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions. Most authors found that the shortcomings arising in traditional perturbation methods can be completely eliminated by these methods.

However, for solving Duffing equation with the third nonlinearity and large nonlinear parameter the solution procedure of *He's parameter expanding method and homotopy perturbation method* are of deceptive simplicity, and the insightful solutions obtained are of high accuracy even for the zero-order approximation. These methods have eliminated limitations of the traditional perturbation methods. On the other hand they can take full advantage of the traditional perturbation techniques, thus there has been a considerable deal of research in applying these techniques for solving various strongly nonlinear equations.

4 Conclusions

In this paper, some new analytical asymptotic methods for solution of a nonlinear problem such as Duffing oscillator equation with very strong nonlinearity are presented. Four different techniques based upon the variational iteration method, the parameterized perturbation method, *He's parameter expanding method* and *homotopy perturbation method* predicts the period of the system in adequate manner. For the nonlinear oscillators, all the reviewed methods yield high accurate approximate periods.

Comparison of the obtained results with those of the exact solution shows that these methods are very effective and convenient and quite accurate to both linear and nonlinear physics and engineering problems. However, the *homotopy perturbation method* in compare with the other methods is relatively easy to apply even for more complex arrangements.

As a result, we conclude that these methods have given very good accuracy in this particular problem of Duffing oscillator and can be easily extended to some other nonlinear problems i.e. Duffing oscillator with forcing term, forcing oscillator with quadratic type damping and Duffing oscillator with excitation term.

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Nomenclature

a	Coefficient
A	Initial condition
b	Coefficient
c	Coefficient
g	Function
L	Linear operator
N	Nonlinear operator
p	Bookkeeping parameter
R	Residual
t	Independent variable
T	Period
v	Variable
u	Unknown parameter of the system
x	Independent variable

Greek symbols

α	Non-zero constant function
ε	Nonlinear parameter
ζ	Variable
λ	General Lagrange multiplier
τ	Variable
ω	Frequency

Subscripts

0	Initial conditions
exa	Exact solution

Figures

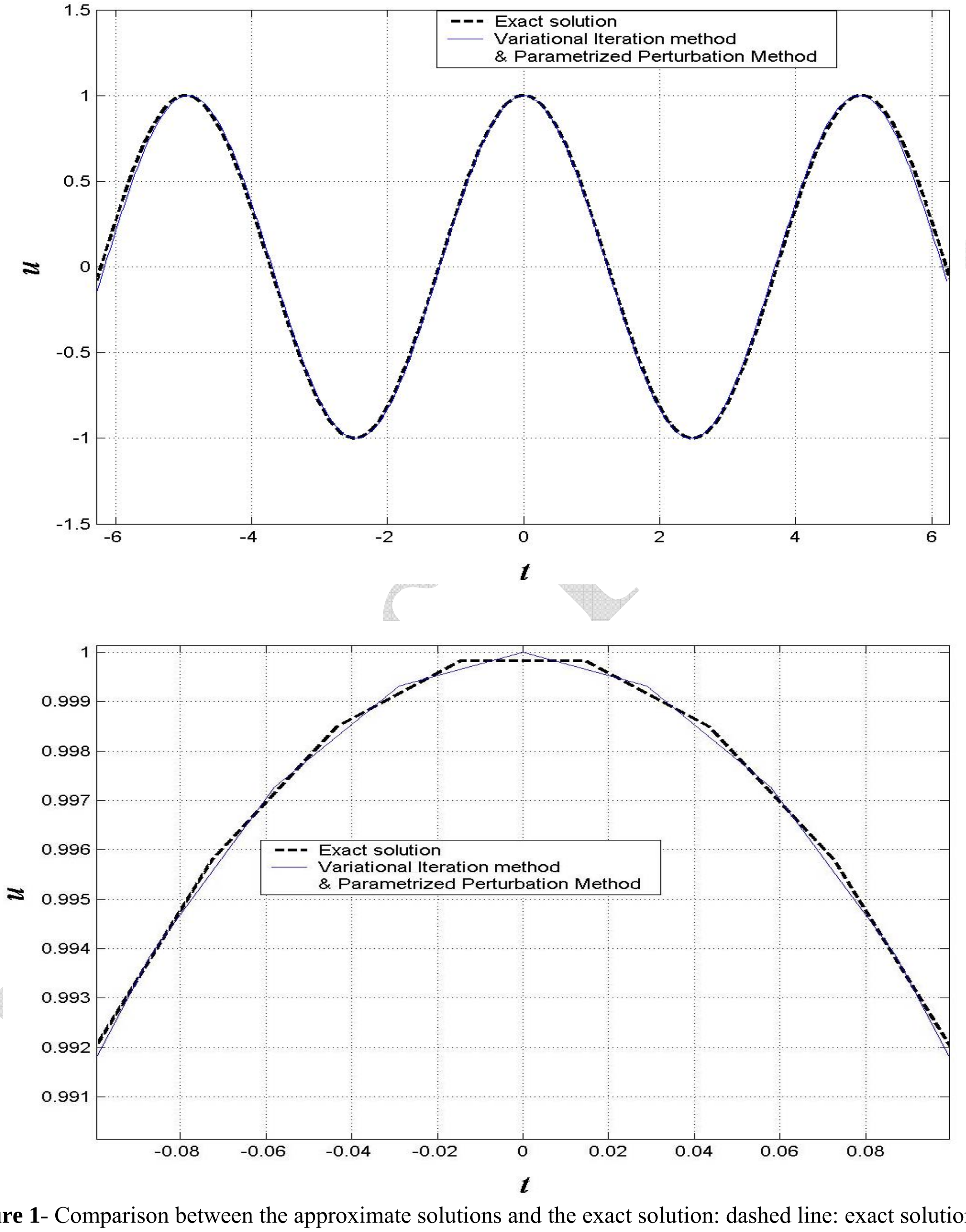

Figure 1- Comparison between the approximate solutions and the exact solution: dashed line: exact solution and solid line: the approximate solutions.

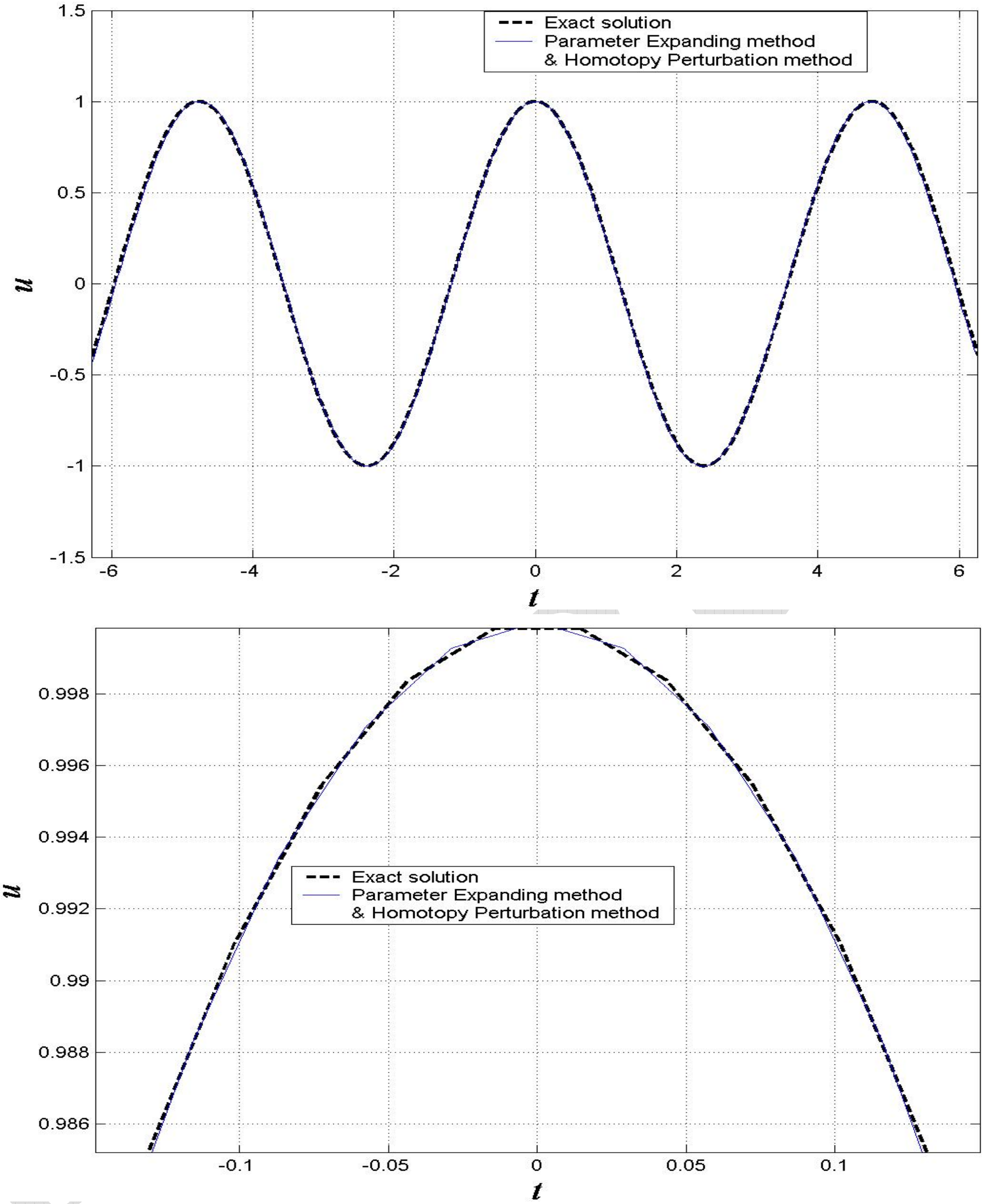

Figure 2- Comparison between the approximate solutions and the exact solution: dashed line: exact solution and solid line: the approximate solutions.

چکیده

در این تحقیق، هدف حل معادله غیرخطي دافینگ با پارامتر غیر خطي بزرگ مي باشد. از ۴ روش مختلف جهت مطالعه معادله حرکت استفاده شده است؛ روش تغییرات تکراري، روش بسط پارامتري He، روش اختلال پارامتري و روش اختلال هموتوپي. با توجه به نتایج، مي توان ادعا کرد تقریب بکاربرده شده در روش هاي مذکور در رنج گسترده اي از پارامترهاي مسئله صحیح است و بنابراین این روش ها ابزار قدرتمندي براي حل انواع سیستم هاي غیرخطي در مسائل مهندسي هستند.