Vibration Analysis of a Rectangular Composite Plate in Contact with Fluid

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In this paper, modal analysis of the fluid-structure interaction has been investigated. Using classical laminated plate theory, a closed form solution for natural frequencies of FSI is extracted. For fluid, homogenous, inviscid and irrotational fluid flow is assumed. Then, a combined governing equation for the plate-fluid system is derived. In order to validate the equations and results, they are compared with results reported in other literatures. The vibration behavior for different plate length to width ratios are also studied. For the forced vibration, three cases; harmonic point load, distributed loading and step pressure loading; are performed and for each case, the time response of plate-fluid system is obtained. Also, frequency response of plate-fluid system has been achieved for harmonic load.

Keywords: modal analysis, natural frequency, mode shape, fluid-structure interaction, composite plate.

1 Introduction

Obtaining the natural frequencies and mode shapes of a structure like beam, plate, etc. due to the presence of fluid is generally known as the fluid-structure interaction (FSI) problem. These problems are covering a broad area of applications in engineering and marine industries, such as the vibration problem of offshores, ship structures, reservoirs, dams and signaling problems of submarines and torpedoes. Nowadays, using the composite plates comparing to other metal alloys, in applications like; civil, astronautic industries, etc. because of the better strength to weight ratio of them is increasingly growing. Thus, a good understanding of the dynamic interaction between an elastic plate and fluid is necessary.
In addition, the existence of fluid around the structure causes the kinetic energy to increase considerably. Consequently, the natural frequencies of the plate coupled with fluid significantly decrease in comparison with those of the plate in the air. Therefore, it is essential to find the natural frequencies of the structures immersed in or in contact with fluid, since the natural frequencies in contact with fluid are different from those in air. Both analytical and numerical methods have been used for FSI problems in literatures. The analytic approaches are restricted to some special cases, and the numerical methods, such as fluid finite element method (FFEM) and boundary element method (BEM) could be used for general cases.

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However, the FFEM and BEM naturally require a huge time for modeling and computations, in addition, there is some difficulties to explain the qualitative effects of fluid. The analytic approach for the addressed problem was initiated by Rayleigh [1] at 1877. He calculated the increase of inertia of a rigid disc vibrating in a circular aperture. Haddara and Cao [2] derived and approximate expression of the modal added masses for cantilever rectangular plate horizontally submerged in fluid, using analytical and experimental data. They also studied the effects of the boundary conditions and submergence depth. Watanabe et al. [3] studied the forced vibration of floating rectangular plates under the moving loads using the FEM. They reported that the depth of fluid has a considerable effect on dynamical response of plate. Zhou and Cheung [4] investigated vibration characteristic of a rectangular plate in contact with fluid on one side, employing the Rayleigh-Ritz approach. In their study, the fluid is filled in a rigid rectangular domain, which has a free surface and is infinite in the length direction. Bermudez et al. [5] using the FEM, studied the free and forced vibration of rectangular plate on incompressible finite fluid. Kerboua and Lakis [6] proposed a semi-analytical method for vibration of pontoon-type plates affected by fluid flow.

First, a solution was initiated by an analytical implicit response for fluid problem; then, another solution for the vibration of plate was found using FEM and final equations was combined as an eigenvalue problem. Khorshidi [7, 8], addressed the problem of linear free vibration of a rectangular thin plate partially in contact with fluid. Natural frequencies and wet mode shapes of the plate coupled with fluid, using the Rayleigh-Ritz method was the results of his works. Hosseini-Hashemi et al. [9] proposed a semi-analytical solution for the free vibration of multi-span, moderately thick, rectangular plate. In their work, the resulting Galerkin equation was solved by application of the Rayleigh-Ritz minimization method. Hosseini-Hashemi et al. [10] studied the free vibration of a rectangular submerging plate for six different boundary conditions. Bakhsheshi and Khorshidi [11], studied the free vibration of a FGM rectangular plate, partially in contact with fluid. Their work was based on the Rayleigh-Ritz method. Rezvaani et al. [12] studied the fluid virtual added mass effect on the natural frequencies of the plate. First, they addressed the problem, analytically, and then, they used some experimental tests and software simulations in ANSYS for validation of their results. Robinson and Palmer [13] performed vibration analysis for a rectangular plate floating on a body of fluid. They derived the transfer function for a harmonic point load, but their analysis is valid only for a finite number of lower frequency modes.

In this paper, free and forced vibration of the rectangular composite CLPT plate, floating on the surface of an inviscid fluid; using the modal analysis expansion method, has been investigated and the natural frequencies and mode shapes of the FSI for the simply supported edges has been extracted. The previous similar works have not focused on the composite plate; therefore, the modal expansion method used for the forced vibration analysis, has been used for the first time here. Similar works concerning the FSI problem have differences in both modeling and solution methodology. Some of the similar works have considered an infinite physical domain for the fluid, while in this paper we considered limited domains. Hence, similar works used different methods to find the solutions, like as Fourier transformation method, etc. while we used the method of modal expansion. The analysis presented in this paper is of a FSI problem in which the plate and the fluid modes are compatible. First, a rectangular composite plate with unspecified edge condition; floating on a body of incompressible fluid, is considered. After addressing the combined governing equation, free motion in the combined mode is investigated, then, the constraints on the mode shapes are developed. Afterward, natural frequencies of composite plate with and without fluid has been calculated and the results are compared with the other works. Also, with consideration of length to width ratio of plate, the effects of aspect ratio have been considered.

Finally, dynamic deflection response of plate for three important cases; concentrated harmonic loading, distributed harmonic pressure loading and step pressure load, using the above mentioned modal expansion method has obtained. Then, frequency response of the FSI has been extracted.

2 Physical Modeling and Formulation

Here we consider the physical model of a horizontal, rectangular, composite plate floating on the surface of a body of liquid, where a, b and h represent the length, width and the thickness of the rectangular plate, respectively. F Denotes the fluid domain and S_{fs} denotes the surface between fluid and the plate. The weight of the plate is assumed to be supported by the buoyancy forces and the dry surface of the plate is under a varying external pressure $p(x_1, x_2, t)$, while the pressure acting on the wet surface is $p'(x_1, x_2, t)$ as shown in figure (1).

The governing equation of the forced vibration of the rectangular orthotropic composite plate in contact with fluid, neglecting the effects of the rotatory inertia and shear deformation effects can be written as [13]:

$$\rho_p h \frac{\partial^2 w}{\partial t^2} + \left(D_{11} \frac{\partial^4}{\partial x_1^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4}{\partial x_1^2 \partial x_2^2} + D_{22} \frac{\partial^4}{\partial x_2^4} \right) w = p'(\mathbf{x}, t) - p(\mathbf{x}, t) \tag{1}$$

Where ρ_p is the mass density of plate and D_{ij}'s are bending stiffness coefficients of the composite plate [14], which are introduced as:

$$\begin{aligned} D_{11} &= \frac{E_{11} h^3}{12(1 - \nu_{12}\nu_{21})} \quad D_{12} = \nu_{21} D_{11} \\ D_{22} &= \frac{E_{22} h^3}{12(1 - \nu_{12}\nu_{21})} \quad D_{66} = G_{12} \frac{h^3}{12} \end{aligned} \tag{2}$$

Property relations for the laminates of the composite, in the case that local and global coordinates does not coincide and have a counterclockwise angle θ, are included in the appendix II.

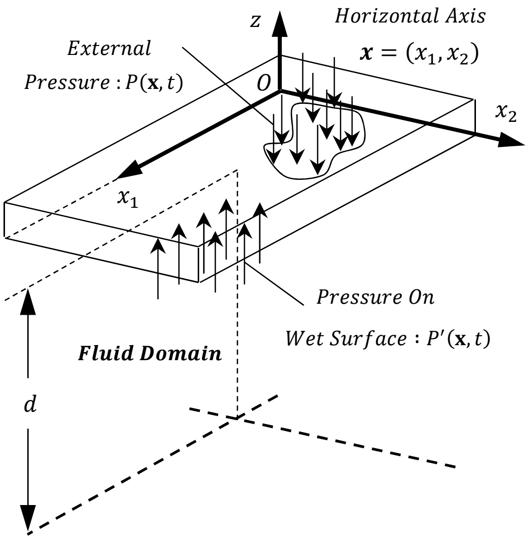

Figure 1 Rectangular composite plate floating on a fluid of constant depth d

The assumptions for dynamic modeling of the fluid are including: (a) fluid is homogeneous and incompressible, (b) low-amplitude oscillations of fluid are assumed and (c) fluid is inviscid and its motion irrotationaly. According to these assumptions of velocity potential $\phi(x_1, x_2, z, t)$, Laplace equation is hold:

$$\nabla^2\phi \triangleq \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{3}$$

The fluid surface condition is derived from the unsteady Bernoulli equation. In this analysis, we consider problems where there is heavy fluid loading, such that the weight of the fluid is significant. However, confining the analysis for low-frequency, low-amplitude oscillations, in which particle velocities are small, the convective inertia terms can be ignored. Thus, the pressure at any point in the fluid domain is presented by, P, where:

$$gz + \frac{P}{\rho_f} - \frac{\partial\phi}{\partial t} = 0 \tag{4}$$

At the surface of the fluid, if we assume that the deflection of the plate is smaller than the fluid depth, this equation becomes:

$$\rho_f g w(x_1, x_2, t) + P'(x_1, x_2, t) - \rho_f \left(\frac{\partial\phi}{\partial t}\right)\Big|_{z=0} = 0 \tag{5}$$

Now, substituting $P'(\mathbf{x}, t)$ from equation (5) in equation (1), partial differential equation governing the forced vibration of a rectangular composite CLPT plate floating on fluid, could be extracted:

$$\rho_p h \frac{\partial^2 w}{\partial t^2} - \rho_f \left(\frac{\partial\phi}{\partial t}\right)\Big|_{z=0} + \left(D_{11}\frac{\partial^4}{\partial x_1^4} + 2(D_{12} + 2D_{66})\frac{\partial^4}{\partial x_1^2 \partial x_2^2} + D_{22}\frac{\partial^4}{\partial x_2^4}\right) w \\ + \rho_f g w = -P(\mathbf{x}, t) \tag{6}$$

At the interaction domain of plate-fluid, if one assume that the displacements are small, this led to equating velocities [13] and we have:

$$-\left(\frac{\partial\phi(\mathbf{x}, z, t)}{\partial z}\right)\Big|_{z=0} = \frac{\partial w(\mathbf{x}, t)}{\partial t} \tag{7}$$

3 Free Vibrations

Here we consider the response of the fluid and the plate in any one combined modes. Separable solutions are assumed for the displacement response and the velocity potential, so that:

$$w(\mathbf{x}, t) = \psi(\mathbf{x})T(t) \\ \phi(\mathbf{x}, z, t) = U(\mathbf{x})F(z)G(t) \tag{9}$$

Where, $\psi(\mathbf{x})$ describes the horizontal spatial variation of w and $T(t)$ is the time variation. $U(\mathbf{x})$, $F(z)$ and $G(t)$ are the horizontal, vertical and temporal variation of ϕ. Then, substituting from Eqs. (8)-(9) in equation (7) and some simplifications would result a new form for the velocity potential $\phi(\mathbf{x}, z, t)$ as below:

$$\phi(\mathbf{x}, z, t) = -\psi(\mathbf{x}) \frac{F(z)}{\left(\frac{dF(z)}{dz}\right)\Big|_{z=0}} \dot{T}(t) \tag{10}$$

and substituting this velocity potential in Laplace equation, results two separate differential equations:

$$\nabla^2\psi(\mathbf{x}) + \mu\psi(\mathbf{x}) = 0 \tag{11}$$

$$\frac{d^2F(z)}{dz^2} - \mu F(z) = 0 \tag{12}$$

Where, the parameter μ in this equations is a constant real number. Those solutions with μ as a complex number, have no physical interpretation in wave theory and we ignore them. Now at the bed of the fluid container, there is no normal component of velocity and Neumann boundary condition holds, $\left(\frac{\partial\phi}{\partial z}\right)\Big|_{z=-d} = 0$. Then using equation (12), one can see:

$$F(z) = c_1\cosh\big(\lambda(z+d)\big), \quad \mu = \lambda^2 \tag{13}$$

3-1 Mode Shapes and Natural Frequencies

For free vibrations in any one combined mode, expressions (8) and (10) are substituted in equation (6) and it is supposed $P(x_1, x_2, t) = 0$, which gives:

$$\big(\rho_p h + m_f\big)\ddot{T}(t) + KT(t) = 0 \tag{14}$$

$$\left(D_{11}\frac{\partial^4}{\partial x_1^4} + 2(D_{12} + 2D_{66})\frac{\partial^4}{\partial x_1^2\partial x_2^2} + D_{22}\frac{\partial^4}{\partial x_1^4}\right)\psi + \big(\rho_f g - K\big)\psi = 0 \tag{15}$$

Where, $m_f = \rho_f\left(\frac{\coth(\lambda d)}{\lambda}\right)$ is added mass effect initiated from the fluid and K is a constant real parameter and will be accurately calculated later. Now considering a separable solution for $\psi(\mathbf{x})$ in Cartesian coordinates, $\psi(\mathbf{x}) = \eta_1(x_1)\eta_2(x_2)$, and applying equation (11), for nontrivial solutions, gives:

$$\frac{d^2\eta_1(x_1)}{dx_1^2} + \gamma_1^2\eta_1(x_1) = 0 \tag{16}$$

$$\frac{d^2\eta_2(x_2)}{dx_2^2} + \gamma_2^2\eta_2(x_2) = 0 \tag{17}$$

Where γ_1 and γ_2 are constant parameters such that, $\gamma_1^2 + \gamma_2^2 = \mu$, which have general solutions:

$$\eta_1(x_1) = a_1\cos(\gamma_1 x_1) + b_1\sin(\gamma_1 x_1) \tag{18}$$

$$\eta_2(x_2) = a_2\cos(\gamma_2 x_2) + b_2\sin(\gamma_2 x_2) \tag{19}$$

Where a_i, b_i $(i = 1{,}2)$ are constants which should be determined by substituting appropriate edge conditions. It should be noted that, mode shapes, unknown constants γ_1 and γ_2 and the promised parameter K would be calculated using the mentioned boundary conditions.

3-2 Simply Supported Boundary Condition

As an example, a rectangular plate $(a \times b)$ floating on a fluid tank $(a \times b \times d)$ is considered, as shown in figure (2).

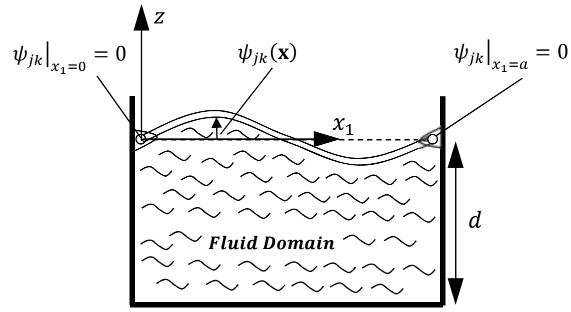

Figure 2 Mode shape and simply supported edge restraints for a rectangular composite plate floating on a rectangular fluid tank

The plate edges are constrained to have zero deflections and are in the mathematical form as:

$$w(\mathbf{x},t)|_{x_1=0,a} = w(\mathbf{x},t)|_{x_2=0,b} = 0 \tag{20}$$

And also, there is no component of fluid velocity normal to the tank walls. Thus,

$$\left.\left(\frac{\partial\phi}{\partial x_1}\right)\right|_{x_1=0,a} = 0 \tag{21-1}$$

$$\left.\left(\frac{\partial\phi}{\partial x_2}\right)\right|_{x_2=0,b} = 0 \tag{21-2}$$

After applying these conditions to Eqs. (18)-(19), leads to finding parameters γ_i as $\gamma_1 = \gamma_{1m} = \frac{m\pi}{a}$ and $\gamma_2 = \gamma_{2n} = \frac{n\pi}{b}$, and the surface displacements mode shapes as:

$$\psi_{mn}(x_1,x_2) = a_{mn}\sin\left(\frac{m\pi}{a}x_1\right)sin\left(\frac{n\pi}{b}x_2\right), \quad m,n = 1,2,3,\ldots \tag{22}$$

These happen to be orthogonal. Normalizing so that, $\iint_A \psi_{mn}\psi_{pq}dx_1dx_2 = A\delta_{mp}\delta_{nq}$, where A is the area of the plate and δ_{mp} is the Kronecker delta function , gives:

$$\psi_{mn}(\mathbf{x}) = \begin{cases} 2\sin\left(\frac{m\pi}{a}x_1\right)sin\left(\frac{n\pi}{b}x_2\right) & m,n \neq 0 \\ 0 & otherwise \end{cases} \tag{23}$$

Now, substituting these mode shapes in to equation (15), leads to finding constant parameter K:

$$K = K_{mn} = \rho_f g + D_{11}\left(\frac{m\pi}{a}\right)^4 + 2(D_{12}+2D_{66})\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^4 \tag{24}$$

And using the equation (15), natural frequencies of the rectangular composite plate in contact with fluid; with simply supported edges, are:

$$\omega_{f_{mn}} = \left[\frac{K_{mn}}{\rho_p h + \rho_f\left(\frac{\coth(\lambda_{mn}d)}{\lambda_{mn}}\right)}\right]^{\frac{1}{2}} \tag{25}$$

3-3 Results and Validation of free vibration

In this section, the analytically found natural frequencies are validated, for the case without of fluid and then, for the FSI case. An important point could be seen in relation (25), that is if we ignore the fluid terms in this equation ($\rho_f = 0$), natural frequencies of a composite plate without fluid are found, separately. In the other words, the fluid affected terms in equation (25), are separate from the plate terms, which are known as virtual added mass terms [12]. Parameters and engineering constants which are used in this research are listed in appendix I.

For the case $\rho_f = 0$ (without fluid), non-dimensional natural frequencies of the composite plate are defined as, $\bar{\omega}_{a_{mn}} = \omega_{a_{mn}} \left(\frac{b}{\pi}\right)^2 \sqrt{\frac{\rho_p h}{D_{22}}}$, and are equal to:

$$\bar{\omega}_{a_{mn}} = \left[\bar{D}_{11}\left(\frac{b}{a}\right)^4 m^4 + \bar{D}_{12}\left(\frac{b}{a}\right)^2 m^2 n^2 + n^4\right]^{\frac{1}{2}} \tag{26}$$

Where $\bar{D}_{11} = \frac{D_{11}}{D_{22}}$ and $\bar{D}_{12} = 2\left(\frac{D_{12}}{D_{22}} + 2\frac{D_{66}}{D_{22}}\right)$ are the non-dimensional stiffness constants of the composite plate. Comparison of this results with references [14] and [15] are presented in tables (1) and (2), respectively.

As one can see in table (1), fundamental non-dimensionalized natural frequencies of a lamina is decreasing with increase in the number of modes. The results of table (2) show that, the natural frequencies of a lamina is greater than a laminated three layer composite, initiating from decreasing of the stiffness to mass ratio.

In the following, if we put $m = n = 1$ in the equation (25), fundamental natural frequencies of composite plate will be calculated. The variation of this frequencies as a function of aspect ratio of plate ($^a/_b$), are shown in figure (3).

Table 1 Non-dimensionalized frequencies of laminated composite plate, $\bar{\omega}_{a_{mn}}$, according to the CLPT plate, in comparison with reference [14]

0^o			
Reference [14]	Present work	(m,n)	E_1/E_2
3.672	3.6674	(1,1)	10
14.690	14.6697	(2,2)	10
33.053	33.0068	(3,3)	10
58.692	58.6788	(4,4)	10
91.701	91.6856	(5,5)	10
4.847	4.8451	(1,1)	20
19.388	19.3804	(2,2)	20
43.623	43.6059	(3,3)	20
77.530	77.5216	(4,4)	20
121.133	121.1275	(5,5)	20

Table 2 Non-dimensionalized frequencies of laminated composite plate, $\bar{\omega}_{a_{mn}}$, according to the CLPT plate, in comparison with reference [15]

$(45^o/-45^o/45^o)$					
$^a/_h = 0.02$		$^a/_h = 0.05$		Mode number	E_1/E_2
Reference [15]	Present work	Reference [15]	Present work		
1.576	1.5782	3.646	3.6575	1	40
4.837	4.9042	13.506	13.5160	2	40
11.227	11.3677	23.001	23.1569	3	40

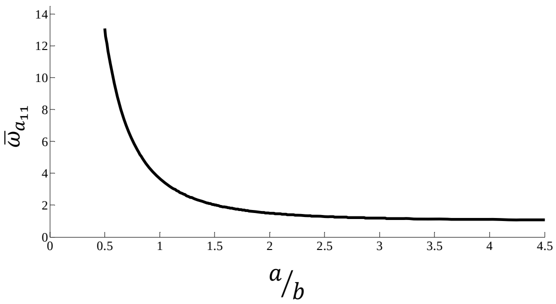

Figure 3 Non-dimensionalized fundamental frequencies, $\overline{\omega}_{a_{11}}$, as a function of plate aspect ratio ($^{a}/_{b}$), for symmetric $(0^o/90^o)_s$ graphite-epoxy laminated plate, with $\frac{E_{11}}{E_{22}} = 10$

In figure (3), as much as the aspect ratio is growing, the values of fundamental natural frequencies of composite plate are considerably decreasing; which means for a plate with constant width, the more the length of plate is larger, the less the value of fundamental natural frequency is.

For the second case ($\rho_f \neq 0$), the square values of fundamental natural frequencies of the FSI system are:

$$\overline{\omega}_{f_{mn}}^2 = \frac{\frac{\rho_f g}{D_{22}}\left(\frac{b}{\pi}\right)^4 + \left(\overline{D}_{11}\left(\frac{b}{a}\right)^4 m^4 + \overline{D}_{12}\left(\frac{b}{a}\right)^2 m^2 n^2 + n^4\right)}{1 + \left(\frac{\rho_f}{\rho_p}\right)\left(\frac{b}{\pi}\right)\frac{1}{h}\frac{\coth\left(d\left(\frac{b}{\pi}\right)^{-1}\sqrt{\left(\frac{b}{a}\right)^2 m^2 + n^2}\right)}{\sqrt{\left(\frac{b}{a}\right)^2 m^2 + n^2}}} \tag{27}$$

Comparing the analytical values of $\overline{\omega}_{f_{mn}}$ from equation (27), with those of reported in reference [16] are shown in table (3). This comparison shows that, addition of fluid to the system causes a considerable decrease in amount of natural frequencies of plate, and also, the differences between the natural frequency values are initiated from approximate solution method used in reference [16], which are in an acceptable level.

In figure (4), non-dimensionalized natural frequencies of FSI in comparison with those of without fluid as a function of aspect ratio are shown.

Table 3 Non-dimensionalized natural frequencies of isotropic plate floating on fluid $\overline{\omega}_{f_{mn}}$, according to the CLPT

Reference [16]	Present work	(m,n)	E_1/E_2
2.3708	2.1689	(1,1)	10
11.106	10.5534	(2,2)	10
26.245	25.9073	(3,3)	10
50.274	48.4198	(4,4)	10
81.045	78.1633	(5,5)	10
3.001	2.8673	(1,1)	20
14.337	13.9522	(2,2)	20
35.184	34.2509	(3,3)	20
67.136	64.0136	(4,4)	20
105.037	103.3361	(5,5)	20

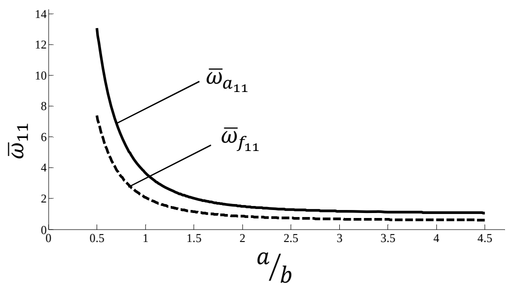

Figure 4 Comparison between non-dimensionalized fundamental frequencies, $\bar{\omega}_{a_{11}}$ and $\bar{\omega}_{f_{11}}$, as plate aspect ratio $^{a}/_{b}$ varies, for symmetric $(0^o/90^o)_s$ graphite-epoxy laminate, with $\frac{E_{11}}{E_{22}} = 10$

It can be interpreted from figure (4), that the natural frequencies of FSI system are considerably smaller than the plate without fluid. The reason of this is that adding fluid to the system, causes increase in kinetic energy of the system and subsequently, it causes decreasing the natural frequencies, and this is why the fundamental natural frequency curve of FSI is below that of the plate without fluid.

4 Forced Vibrations

In section 3, general solution of the plate displacement and velocity potential function of fluid, in any one combined mode, were analytically calculated. Now, the modal expansion of the displacement and velocity potential could be presented in the form of:

$$w(\mathbf{x},t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \psi_{jk}(\mathbf{x}) T_{jk}(t) \tag{28}$$

$$\phi(\mathbf{x},z,t) = -\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \psi_{jk}(\mathbf{x}) \frac{\cosh\left(\lambda_{jk}(z+d)\right)}{\lambda_{jk}\sinh\left(\lambda_{jk}d\right)} \dot{T}_{jk}(t) \tag{29}$$

Where, λ_{jk} equals to $\lambda_{jk} = \sqrt{\left(\frac{j\pi}{a}\right)^2 + \left(\frac{k\pi}{b}\right)^2}$.

Now, substituting expressions (28) and (29) into equation (6), and then multiplying the resulted equation by $\psi_{jk}(\mathbf{x})$ and integrating over the surface of the plate, A, gives:

$$\iint_A \psi_{jk}(\mathbf{x}) \left\{ \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \left[M_{jk}\ddot{T}(t) + K_{jk}T(t)\right]\psi_{jk}(\mathbf{x}) \right\} dx_1 dx_2 = -\iint_A \psi_{jk}(\mathbf{x})\{P(\mathbf{x},t)\} dx_1 dx_2 \tag{30}$$

Where, $M_{jk} = \left(\rho_p h + \rho_f \frac{\coth(\lambda_{jk}d)}{\lambda_{jk}}\right)$.

It is possible to simplify the analysis at this stage by assuming that the mode shapes of the liquid loaded plate are orthogonal. Those given in equation (23) happen to be orthogonal, but in general, this is not so. However, for a sufficiently thin plate with simply supported edges, oscillating at low frequency, the motion is dominated by the liquid. Liquid-coupled plate modes can then be approximated by the surface response of a body of liquid with a smooth and continuous surface. For problems of this nature, the normalization condition of the plate $\iint_A \psi_{jk}(\mathbf{x})\psi_{qr}(\mathbf{x}) = A\delta_{jk}\delta_{qr}$ is assume to hold. Thus, equation (30) becomes a series of decoupled modal expressions, which could be used for finding time responses.

4-1 Time Response

In this section, time response of the forced vibrations of FSI system in any position, for three case of (a) unit harmonic external excitation with frequency of ω applied at point $\mathbf{x_0}$, (b) distributed harmonic pressure excitation and (c) distributed unit step pressure excitation, will be calculated.

For the case (a), external force applied to the FSI equals to, $P(\mathbf{x},t) = \delta(\mathbf{x}-\mathbf{x_0})\sin(\omega t)$ and if we assume that the plate has been initially at rest, initial values of the system could be considered as $T(0) = \dot{T}(0) = 0$. However, dynamical steady state deflection of the forced vibration of FSI system, in any one combined mode, is:

$$w(\mathbf{x},t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} L_{jk}\sin\left(\frac{j\pi x_1}{a}\right)\sin\left(\frac{k\pi x_2}{b}\right)\sin(\omega t) \tag{31}$$

Where, $L_{jk} = \frac{-2\psi_{jk}(\mathbf{x_0})}{ab\{K_{jk}-\omega^2 M_{jk}\}}$. Dynamic response curve of the steady state forced vibration of the FSI, at point $\mathbf{x_0} = \left(\frac{a}{2},\frac{b}{2}\right)$, is shown in figure (5).

Displacement as a function of time in figure (5), shows a harmonic behavior with frequency $\omega = 5^{\frac{rad}{sec}}$. Also, considering mode numbers, $m = n = 50$, amplitude of vibrations in equation (31), converges to $0.38021^{\mu m}$.

For the case (b), external force applied is $P(\mathbf{x},t) = P_0\sin(\omega t)$, with $P_0 = 0.5^{kPa}$ and initial values are the same as case (a). Then, dynamical steady state deflection of the forced vibration of FSI system, in any one combined mode, is:

$$w(\mathbf{x},t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} R_{jk}\sin\left(\frac{j\pi x_1}{a}\right)\sin\left(\frac{k\pi x_2}{b}\right)\sin(\omega t) \tag{32}$$

Where $R_{jk} = \frac{-4P_0 S_{jk}}{jk\pi^2\{K_{jk}-\omega^2 M_{jk}\}}$ and $S_{jk} = \left[(-1)^{j+k} - (-1)^j - (-1)^k + 1\right]$. In the following, dynamic response of the steady state forced vibration, at $\mathbf{x_0} = \left(\frac{a}{2},\frac{b}{2}\right)$, is shown in figure (6).

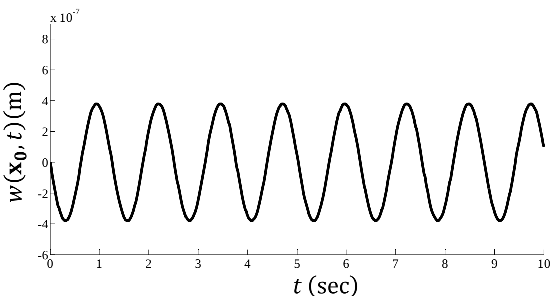

Figure 5 Dynamical response to harmonic point load $F(t) = P_o\sin(\omega t)$, for a symmetric $(0^o/90^o)_s$ graphite-epoxy laminate composite floating on fluid, $\frac{E_{11}}{E_{22}} = 10$

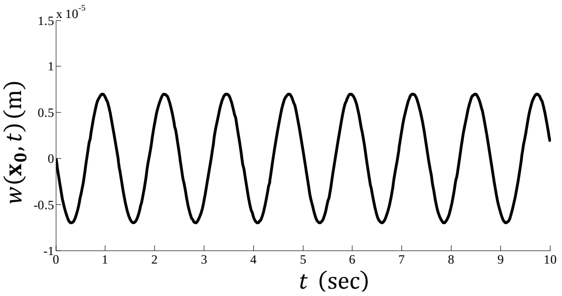

Figure 6 Dynamical response to harmonic distributed force $F(t) = P_o \sin(\omega t)$, for a symmetric $(0^o/90^o)_s$ graphite-epoxy laminate composite floating on fluid, $\frac{E_{11}}{E_{22}} = 10$

This figure shows a harmonic behavior with the frequency $\omega = 5\frac{rad}{sec}$. Also, considering mode numbers $m = n = 50$, amplitude of vibrations in equation (32), converges to $6.986^{\mu m}$.

For the case (c), external force applied is $P(\mathbf{x}, t) = P_0 u(t-0)$, with $P_0 = 0.5^{kPa}$ and initial values are the same as the previous cases. Then, dynamical steady state deflection of the forced vibration of FSI system, in any one combined mode, is:

$$w(\mathbf{x}, t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} H_{jk} \sin\left(\frac{j\pi x_1}{a}\right) \sin\left(\frac{k\pi x_2}{b}\right) \left(1 - \cos\left(\omega_{f_{jk}} t\right)\right) \tag{33}$$

Where H_{jk} is $\frac{-4P_0 S_{jk}}{jk\pi^2 K_{jk}}$. Dynamic response of the steady state forced vibration, is shown in figure (7). This figure shows a harmonic behavior and despite two previous cases, frequencies of vibration are equal to natural frequencies of the FSI system.

4-2 Frequency Response

In this section, first we apply an external concentrated excitation to the center of the plate and then, the Domain-Frequency response will be found. For this purpose, equation (30) should be written as a series of decoupled modes as:

$$M_{jk}\ddot{T}(t) + K_{jk}T(t) = -\frac{\iint_A \psi_{jk} P(\mathbf{x}, t) dx_1 dx_2}{A} \tag{34}$$

Consider now the time response at $\mathbf{x}$ to an unit, harmonic, point load at $\mathbf{x_0}$, of frequency ω. Then $P(\mathbf{x}, t) = e^{i\omega t}\delta(\mathbf{x} - \mathbf{x_0})$ and substituting in equation (34), forced vibration equation for any decoupled mode would be resulted as:

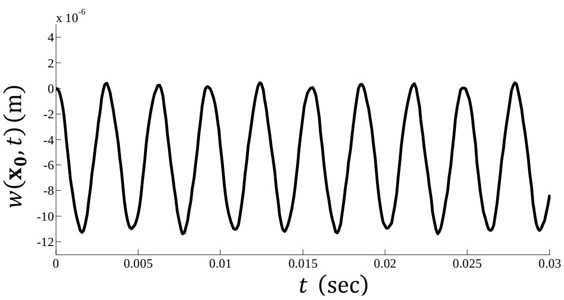

Figure 7 Dynamical response to harmonic distributed force $F(t) = P_o \sin(\omega t)$, for a symmetric $(0^o/90^o)_s$ graphite-epoxy laminate composite floating on fluid, $\frac{E_{11}}{E_{22}} = 10$

$$M_{jk}\ddot{T}(t) + K_{jk}T(t) = -\left(\frac{\psi_{jk}(\mathbf{x})}{A}\right)e^{i\omega t} \tag{35}$$

If it is assumed that $T(t) = Y_{jk}(\omega)e^{i\omega t}$, then substituting in equation (35), $Y_{jk}(\omega)$ could be found:

$$Y_{jk}(\omega) = \frac{-\psi_{jk}(\mathbf{x_0})}{A\left(K_{jk} - \omega^2 M_{jk}\right)} \tag{36}$$

So, from equation (28), the response at $\mathbf{x}$ to an unit, harmonic, point loaded at $\mathbf{x_0}$ could be found. On the other hand, the displacement could be written as $w(\mathbf{x}, t) = G(\mathbf{x}, \mathbf{x_0}, \omega)e^{i\omega t}$, where $G(\mathbf{x}, \mathbf{x_0}, \omega)$ is the frequency domain transfer function. Now, if we show the transfer function as $G(\mathbf{x}, \mathbf{x_0}, \omega) = \psi_{jk}(\mathbf{x})H_{jk}(\omega)\psi_{jk}(\mathbf{x_0})$, the domain $H_{jk}(\omega)$ as a function of frequency ω is:

$$H_{jk}(\omega) = \frac{-1}{AK_{jk}\left(1 - \left(\frac{\omega}{\omega_{jk}}\right)^2\right)} \tag{37}$$

Drawing the frequency domain $H_{jk}(\omega)$ for a specific range of frequency ω, domain-frequency curve would be resulted as figure (8). As we can interpret from figure (8), frequency domain curve have some peaks at the place of points shown with numbers 1 to 5 and each of this peaks are showing a natural frequency of the system. For the case without fluid, these peaks happen sooner, meaning that, the values of natural frequency of the FSI are lower than those of without fluid. The frequency domain analysis is important, because when dealing with the instability or happening of natural frequencies, it is hard to find any explanation by time domain analysis and, hence, the explanation turned to be very logical when one uses the frequency analysis and the peaks are initiated from presence of natural frequencies. Actually, it may not have any special physical meaning but we could use it as a tool for finding and even comparing natural frequencies of the plate with or without fluid. As one can see in figure (8), for the case without fluid, natural frequencies (peaks) happen sooner, meaning that the values of the natural frequency of FSI are lower than those of without fluid. The more the fluid depth d is increasing, the less the amplitude $H_{jk}(\omega)$ will increase. As a result, we can say that increasing the fluid depth d causes an increase in the natural frequency, hence, it causes a delay in happening of the peaks in figure (8). The frequency response function for the plate supported by liquid is similar to the standard result obtained for the case without fluid, with added mass and stiffness, $\rho_f g$ and $\rho_f\left(\frac{\coth(\lambda_{jk}d)}{\lambda_{jk}}\right)$, respectively. This is equivalent to adding a layer of liquid of depth $\left(\frac{\coth(\lambda_{jk}d)}{\lambda_{jk}}\right)$, moving with the plate. As the liquid level is deeper, the thickness of the layer is approaching $\frac{1}{\lambda_{jk}}$. As the mode number increases, the effect of this layer decreases.

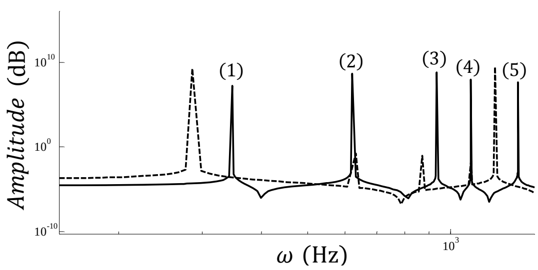

Figure 8 Frequency response, amplitude as a function of frequency plot for composite plate floating on fluid (dash line) and without fluid (solid line)

5 Conclusions

The main objective of this paper is the analytical free and forced vibration analysis of the composite plate floating on fluid. For this purpose, natural frequencies and mode shapes for free vibration were obtained, then, frequency response of forced vibration was calculated, analytically. As a result of this study, it was found that, increasing the number of lamina in composite plate, causes a considerable decrease in the natural frequencies. Also, the more the aspect ratio of the plate is, the less the fundamental natural frequencies of FSI are. Natural frequencies of a plate in presence of fluid are lower than those of without fluid. Forced vibration response of FSI, with external load of type harmonic concentrated force and harmonic distributed pressure, have vibration frequency equal to excitation frequency. Although, for the case of step loading, FSI oscillates with natural frequency of system. Adding fluid under the plate in FSI, causes an increase in mass and stiffness of the plate.

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Nomenclatures

a: length of the plate
b: width of the plate
d: depth of the fluid tank
E_{ij}: Young modulus
G_{ij}: shear modulus
g: gravitational acceleration
h: thickness of the plate

Greek Symbols
ν_{ij}: Poison ratio
ρ_f: fluid mass density
ρ_p: plate mass density

Appendix I

Table 4 Values of parameters and engineering properties used for validation of results

Unit	Values	Engineering constant or parameter
kgm^3	1000	ρ_f
kgm^3	2440	ρ_P
M	0.5	a
M	0.2	b
M	0.01	h
M	2	d
ms^{-2}	9.81	g
-	0.25	ν_{12}
Nm^{-2}	$260x10^9$	E_{11}
Nm^{-2}	$13\ x10^9$, $26\ x10^9$	E_{22}
Nm^{-2}	$0.5E_{22}$	G_{12}
Nm^{-2}	$0.5E_{22}$	G_{13}
Nm^{-2}	$0.2E_{22}$	G_{23}

Appendix II

Let (x, y, z) denotes the global coordinate system used to write the governing equations of a laminate, and let (x_1, x_2, x_3) the local material coordinates of a typical layer of composite plate in the laminate such that x_3-axis is parallel to z-axis (i.e., the x_1x_2-plane and the xy-plane are parallel) and x_1-axis is oriented at an angle of $+\theta$ counterclockwise from the x-axis. see figure (9).

The relations between the generalized plane stress coefficients for the k-th layer; $\bar{Q}_{ij}^{\{k\}}$, and plane stress-reduced stiffnesses; $Q_{ij}^{\{k\}}$, are as below:

$$\bar{Q}_{11}^{\{k\}} = Q_{11}\cos^4(\theta) + 2(Q_{12} + 2Q_{66})\sin^2(\theta)\cos^2(\theta) + Q_{22}\sin^4(\theta)$$

$$\bar{Q}_{12}^{\{k\}} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2(\theta)\cos^2(\theta) + Q_{12}(\sin^4(\theta) + \cos^4(\theta))$$

$$\bar{Q}_{22}^{\{k\}} = Q_{11}\sin^4(\theta) + 2(Q_{12} + 2Q_{66})\sin^2(\theta)\cos^2(\theta) + Q_{22}\cos^4(\theta)$$

$$\bar{Q}_{66}^{\{k\}} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2(\theta)\cos^2(\theta) + Q_{66}(\sin^4(\theta) + \cos^4(\theta))$$

And the relations between $Q_{ij}^{\{k\}}$ and engineering properties of the composite material are:

$$Q_{11}^{\{k\}} = \frac{E_{11}^{\{k\}}}{\left(1 - \nu_{12}^{\{k\}}\nu_{21}^{\{k\}}\right)} \qquad Q_{22}^{\{k\}} = \frac{E_{22}^{\{k\}}}{\left(1 - \nu_{12}^{\{k\}}\nu_{21}^{\{k\}}\right)}$$

$$Q_{12}^{\{k\}} = \frac{\nu_{21}^{\{k\}}E_{11}^{\{k\}}}{\left(1 - \nu_{12}^{\{k\}}\nu_{21}^{\{k\}}\right)} \qquad Q_{66}^{\{k\}} = G_{12}^{\{k\}}$$

And also, bending stiffness coefficients of composite plate; D_{ij}, are presented as:

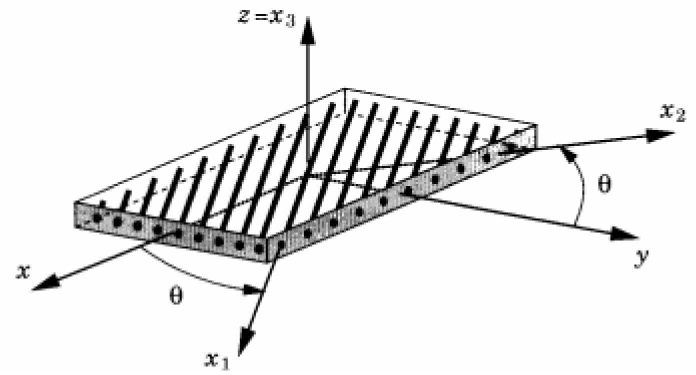

Figure 9 A lamina with local and global coordinates systems [14]

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N}\left\{\bar{Q}_{ij}^{\{k\}}(z_{k+1}^3 - z_k^3)\right\} \quad (i, j = 1,2,6)$$

چکیده

در این مقاله، آنالیز مودال برای مسئله تداخل سازه- سیال مورد بررسی قرار گرفته است. به وسیله تئوری کلاسیک ورق‌های لایه‌ای، یک پاسخ بسته تحلیلی به منظور محاسبه فرکانس‌های طبیعی و شکل مودهای سیستم سازه- سیال استخراج گردیده است. در مدل‌سازی دینامیکی سیال؛ عمده فرضیات صورت گرفته شامل: همگنی، غیرویسکوزی و غیرچرخشی بودن جریان سیال می‌باشد. در ادامه؛ معادله دیفرانسیل مشتق جزئی حاکم بر ارتعاش سیستم ترکیبی سازه- سیال به دست آمده است.
به منظور اعتبارسنجی صحت معادلات و نتایج به دست آمده، به مقایسه آن‌ها با نتایج دیگر مقالات پرداخته شده است. آن‌گاه تاثیر نسبت ابعادی ورق کامپوزیتی بر فرکانس طبیعی سیستم بررسی شده است. برای ارتعاش اجباری؛ سه حالت بارگذاری مختلف شامل بار متمرکز هارمونیک، گسترده هارمونیک و بار فشاری پله‌ای، در نظر گرفته شده و در هر حالت پاسخ زمانی آن‌ها ترسیم گردیده و مورد بررسی قرار گرفته است. هم‌چنین، پاسخ فرکانسی سیستم ورق- سیال تحت بارگذاری هارمونیک استخراج شده است.