Hollow Piezoelectric Cylinder under Transient Loads

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In this paper, transient solution of two dimensional asymmetric thermal and mechanical stresses for a hollow cylinder made of piezoelectric material is developed. Transient temperature distribution, as function of radial and circumferential directions and time with general thermal boundary-conditions, is analytically obtained, using the method of separation of variables and generalized Bessel function. The results are the sum of transient and steady state solutions that depend upon the initial condition for temperature and heat source, respectively. The general form of thermal and mechanical boundary conditions is considered on the piezoelectric cylinder. Material properties of piezoelectric cylinder are the same along the thickness. A direct method is used to solve the Navier equations, using the Euler equation and complex Fourier series.

Keywords: Transient, Thermoelasticity, Hollow Cylinder, Piezoelectric

1 Introduction

Piezoelectric materials are widely used due to their direct and inverse effects. The use of piezoelectric layers as distributed sensors and actuators in structures to control noise and deformations and suppress vibrations is quite common. Several research works have been contributed to model and investigate the basic structural responses of piezoelectric materials i.e. in the pioneering researches of Tiersten [10]. Alashti et al. [1] carried out three-dimensional thermo-elastic analysis of a functionally graded cylindrical shell with piezoelectric layers by differential quadrature method. Alibeigloo and Chen [2] obtained the elasticity solution for an FGM cylindrical panel integrated with piezoelectric layers.
Chu and Tzou [3] presented the transient response of a composite finite hollow cylinder heated by a moving line source on its inner boundary and cooled convectively on the exterior boundary using eigen function expansion method and the Fourier series. Fesharaki et al. [4] presented 2D solution for electro-mechanical behavior of functionally graded piezoelectric hollow cylinder. By using the separation of variables method and complex Fourier series, the Navier equations in term of displacements are derived and solved.

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He et al. [5] derived the active control of FGM plates with integrated piezoelectric sensors and actuators. Mohazzab [6] presents the analytical solution of one-dimensional mechanical and thermal stresses for a hollow cylinder made of functionally graded material. Vaghari et al. [7] presented an analytical method to obtain the transient thermal and mechanical stresses in a functionally graded hollow cylinder subjected to the two-dimensional asymmetric loads. Khoshgoftar et al. [8] presented the thermoelastic analysis of a thick walled cylinder made of functionally graded piezoelectric material by using the separation of variables. Poultangari et al. [9] presented a solution for the functionally graded hollow spheres under nonaxisymmetric thermomechanical loads.
This paper present an analytical method to obtain the transient thermal and mechanical stresses in a piezoelectric hollow cylinder subjected to the two-dimensional asymmetric loads. Temperature distribution is assumed to be a function of radial and circumferential directions and time. The Navier equations are solved analytically using a direct method of series expansion.

2 Governing equation

2.1 Stress distribution

Consider a piezoelectric hollow cylinder of inner radius and outer radius b. Asymmetric cylindrical coordinates (r, θ) are considered along the radial and circumferential directions, respectively. The governing two-dimensional strain-displacement relations in cylindrical coordinates and electric field-electric potential relations are

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} \qquad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right)$$
$$E_r = -\frac{\partial \psi}{\partial r} \qquad E_\theta = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} \tag{1}$$

in which u, v, and ψ as displacement components along the radial and circumferential directions, and the electric potential, respectively. The constitutive relations describing the electrical and the mechanical interaction for the piezoelectric material are

$$\begin{aligned}
\sigma_{rr} &= C_{11}\varepsilon_{rr} + C_{12}\varepsilon_{\theta\theta} - e_{11}E_r - \alpha_r T(r,\theta,t) \\
\sigma_{\theta\theta} &= C_{12}\varepsilon_{rr} + C_{22}\varepsilon_{\theta\theta} - e_{21}E_r - \alpha_\theta T(r,\theta,t) \\
\sigma_{r\theta} &= C_{44}\gamma_{r\theta} - e_{24}E_\theta \\
D_r &= e_{11}\varepsilon_{rr} + e_{21}\varepsilon_{\theta\theta} + \eta_{11}E_r + P_r T(r,\theta,t) \\
D_\theta &= e_{24}\gamma_{r\theta} + \eta_{22}E_\theta + P_\theta T(r,\theta,t)
\end{aligned} \tag{2}$$

Where σ_{ij}, $\varepsilon_{ij}(i,j = r,\theta)$ and $T(r,\theta,t)$ are the stress, strain tensors and the temperature distribution C_{ij} and e_{ij}, η_{ij}, D_i, P_i and α_i are elastic and piezoelectric coefficients, dielectric constants, electric displacements, pyroelectric constant and thermal modulus respectively for the piezoelectric material. The equilibrium equations in the radial and circumferential directions, disregarding the body forces and inertia terms and equation of electrostatic are

$$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}+\frac{1}{r}(\sigma_{rr}-\sigma_{\theta\theta})=0\\
&\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{2}{r}\sigma_{r\theta}=0\\
&\frac{\partial D_{rr}}{\partial r}+\frac{1}{r}\frac{\partial D_{\theta\theta}}{\partial \theta}+\frac{1}{r}D_{rr}=0
\end{aligned} \tag{3}$$

2.2 Heat Conduction Problem

The heat conduction equation in two-dimensional problem for piezoelectric cylinder leads to

$$\dot{T}-\frac{k}{\rho c}\left[T_{,rr}+\frac{1}{r}T_{,r}+\frac{1}{r^2}T_{,\theta\theta}\right]=\frac{R(r,\theta,t)}{\rho c} \tag{4}$$

A comma denotes partial differentiation with respect to the space variable. The symbol dot $(\cdot)$ denotes derivative with respect to time. The initial and mixed boundary conditions are

$$X_{11}T(a,\theta,t)+X_{12}T_{,r}(a,\theta,t)=g_1(\theta,t) \tag{5-a}$$

$$X_{21}T(b,\theta,t)+X_{22}T_{,r}(b,\theta,t)=g_2(\theta,t) \tag{5-b}$$

$$T(r,\theta,0)=g_3(r,\theta) \tag{5-c}$$

where $X_{ij}(i,j=1,2)$ are Robin-type constants related to the thermal boundary condition parameters, and $g_3(r,\theta)$ is the known initial condition. The solution of the heat conduction equations for temperature distribution in piezoelectric cylinder may be assumed to be of the form

$$T(r,\theta,t)=W(r,\theta,t)+Y(r,\theta,t) \tag{6}$$

where $W(r,\theta,t)$ is considered in a way that the boundary conditions of $Y(r,\theta,t)$ become zero. Thus $W(r,\theta,t)$ is assumed a second order polynomial as

$$W(r,\theta,t)=A(t,\theta)r^2+B(t,\theta)r \tag{7}$$

Substituting Eq. (7) into Eqs. (5-a), (5-b) yields

$$A(X_{11}a^2+2X_{12}a)+B(X_{11}a+X_{12})=g_1(\theta,t) \tag{8}$$

$$A(X_{21}b^2+2X_{22}b)+B(X_{21}b+X_{22})=g_2(\theta,t) \tag{9}$$

Substituting Eq. (7) into Eqs. (5-a), (5-b) yields

$$\begin{aligned}
A(\theta,t)&=\frac{g_1(\theta,t)(X_{21}b+X_{22})-g_2(\theta,t)(X_{11}a+X_{12})}{(X_{11}a^2+2aX_{12})(X_{21}b+X_{22})-(X_{11}a+X_{12})(X_{21}b^2+2bX_{22})}\\
B(\theta,t)&=\frac{g_2(\theta,t)(X_{11}a^2+2aX_{12})-g_1(\theta,t)(X_{21}b^2+2bX_{22})}{(X_{11}a^2+2aX_{12})(X_{21}b+X_{22})-(X_{11}a+X_{12})(X_{21}b^2+2bX_{22})}
\end{aligned} \tag{10}$$

and substituting Eqs. (6), (7) into the heat conduction equation Eq. (4) yield

$$\dot{Y}-\frac{k}{\rho c}\left[Y_{,rr}+\frac{1}{r}Y_{,r}+\frac{1}{r^2}Y_{,\theta\theta}\right]=R_1 \tag{11}$$

Where

$$R_1 = \frac{R(r,\theta,t)}{\rho c} - A_{,t} r^2 - B_{,t} r + \frac{k}{\rho c}\left[4A + A_{,\theta\theta} + \frac{1}{r}(B + B_{,\theta\theta})\right] \tag{12}$$

The solution of Eqs. (11) may be obtained by the method of separation of variables. For the general solution, Eq. (13) is substituted into Eq. (11) and using the generalized Bessel function leads to

$$Y(r,\theta,t) = \sum_{n=-\infty}^{n=+\infty}\sum_{m=1}^{\infty} F_{mn}(r) G_{mn}(t) e^{in\theta} \tag{13}$$

$$F_{mn}(r) = C_{mn}(\lambda_{mn} r) \tag{14}$$

Where $F_{mn}(r)$ is derived from the general solution of energy equation without heat source and Substituting Eq. (13) into the Eq. (11) yields

$$G_{mn}(t) = e^{-\int \tau dt}\left[b_{mn} + \int \frac{R^*(t)}{2\pi \|C_{mn}(\lambda_{mn} r)\|^2} e^{\int \tau dt} dt\right] \tag{15}$$

where $\tau = \frac{k}{\rho c}\lambda_{mn}^2$ and $\|C_{mn}(\lambda_{mn} r)\|$ is the norm of the cylindrical function as

$$\|C_{mn}(\lambda_{mn} r)\|^2 = \int_a^b \left[C_{mn}(\lambda_{mn} r)\right]^2 r\, dr \tag{16}$$

$$R^*(t) = \int_0^{2\pi}\int_a^b r\, R_1 C_{mn}(\lambda_{mn} r) e^{-in\theta} dr\, d\theta \tag{17}$$

and b_{mn} is derived from the initial thermal boundary condition defined by Eq. (5-c) as

$$b_{mn} = \frac{1}{2\pi \|C_{mn}(\lambda_{mn} r)\|^2}\int_0^{2\pi}\int_a^b r\left[g_3(r,\theta) - W(r,\theta,0)\right] C_{mn}(\lambda_{mn} r) e^{-in\theta} dr\, d\theta - G^*(0) \tag{18}$$

$$G^*(t) = \int \frac{R^*(t)}{2\pi \|C_{mn}(\lambda_{mn} r)\|^2} e^{\int \tau dt} dt \tag{19}$$

where $C_{mn}(\lambda_{mn} r)$ is the mathematical Cylindrical Function given by

$$C_{mn}(\lambda_{mn} r) = J_n(\lambda_{mn} r) + c_{mn} Y_n(\lambda_{mn} r) \tag{20}$$

$$c_{mn} = -\frac{X_{11} J_n(\lambda_{mn} a) + X_{12} J_n'(\lambda_{mn} a)}{X_{11} Y_n(\lambda_{mn} a) + X_{12} Y_n'(\lambda_{mn} a)} \tag{21}$$

Here, J_n is the Bessel function of the first kind of order n, Y_n is the Bessel function of the second kind of order n, the symbol (') denotes derivative with respect to r, and the eigenvalue λ_{mn} is the positive roots of

$$\begin{gathered}\left[X_{11} J_n(\lambda_{mn} a) + X_{12} J_n'(\lambda_{mn} a)\right] Y_n(\lambda_{mn} b) - \left[X_{11} Y_n(\lambda_{mn} a) + X_{12} Y_n'(\lambda_{mn} a)\right] J_n(\lambda_{mn} b) = 0 \\ m = 1,2,3,\ldots\end{gathered} \tag{22}$$

3 Solution of the problem

Using the relations (1), (2) and (3) the Navier equations in term of the displacements are

$$\begin{aligned}
&u_{,rr}+\frac{1}{r}u_{,r}-\frac{C_{22}}{C_{11}}\frac{1}{r^2}u+\frac{C_{44}}{C_{11}}\frac{1}{r^2}u_{,\theta\theta}+\frac{C_{12}+C_{44}}{C_{11}}\frac{1}{r}v_{,r\theta}-(\frac{C_{44}+C_{22}}{C_{11}})\frac{1}{r^2}v_{,\theta}+\frac{e_{11}}{C_{11}}\psi_{,rr}\\
&+\frac{e_{11}-e_{21}}{C_{11}}\frac{1}{r}\psi_{,r}+\frac{e_{24}}{C_{11}}\frac{1}{r^2}\psi_{,\theta\theta}=\frac{\alpha_r}{C_{11}}T_{,r}+\frac{\alpha_r-\alpha_\theta}{C_{11}}\frac{1}{r}T\\
&v_{,rr}+\frac{1}{r}v_{,r}-\frac{1}{r^2}v+\frac{C_{22}}{C_{44}}\frac{1}{r^2}v_{,\theta\theta}+\left(1+\frac{C_{12}}{C_{44}}\right)\frac{1}{r}u_{,r\theta}+\left(1+\frac{C_{22}}{C_{44}}\right)\frac{1}{r^2}u_{,\theta}+\frac{e_{24}+e_{21}}{C_{44}}\frac{1}{r}\psi_{,r\theta}\\
&+\frac{e_{24}}{C_{44}}\frac{1}{r^2}\psi_{,\theta}=\frac{\alpha_\theta}{C_{44}}\frac{1}{r}T_{,\theta}\\
&u_{,rr}+\left(1+\frac{e_{21}}{e_{11}}\right)\frac{1}{r}u_{,r}+\frac{e_{24}}{e_{11}}\frac{1}{r^2}u_{,\theta\theta}+\left(\frac{e_{21}+e_{24}}{e_{11}}\right)\frac{1}{r}v_{,r\theta}-\frac{e_{24}}{e_{11}}\frac{1}{r^2}v_{,\theta}-\frac{\eta_{11}}{e_{11}}\psi_{,rr}-\frac{\eta_{11}}{e_{11}}\frac{1}{r}\psi_{,r}\\
&-\frac{\eta_{22}}{e_{11}}\frac{1}{r^2}\psi_{,\theta\theta}=-\frac{P_r}{e_{11}}T_{,r}\frac{\partial T}{\partial r}-\frac{P_r}{e_{11}}\frac{1}{r}T-\frac{P_\theta}{e_{11}}\frac{1}{r}T_{,\theta}
\end{aligned}\tag{23}$$

To solve the Navier equations (23) consider the complex Fourier series for displacements $u(r,\theta,t)$ and $v(r,\theta,t)$ and electric potential $\psi(r,\theta)$ as

$$u(r,\theta,t)=\sum_{n=-\infty}^{n=+\infty}\sum_{m=1}^{\infty}u_{mn}(r,t)e^{in\theta}\quad v(r,\theta,t)=\sum_{n=-\infty}^{n=+\infty}\sum_{m=1}^{\infty}v_{mn}(r,t)e^{in\theta}\quad \psi(r,\theta,t)=\sum_{n=-\infty}^{n=+\infty}\sum_{m=1}^{\infty}\psi_{mn}(r,t)e^{in\theta}\tag{24}$$

Substituting Eqs. (24) into Eqs. (23) yield

$$\begin{aligned}
&\sum_{n=-\infty}^{n=+\infty}\sum_{m=1}^{\infty}\left\{u''_{mn}+\frac{1}{r}u'_{mn}-\left(\frac{C_{22}+n^2C_{44}}{C_{11}}\right)\frac{1}{r^2}u_{mn}+\frac{in(C_{12}+C_{44})}{C_{11}}\frac{1}{r}v'_{mn}-\frac{in(C_{44}+C_{22})}{C_{11}}\frac{1}{r^2}v_{mn}\right.\\
&\left.+\frac{e_{11}}{C_{11}}\psi''_{mn}+\frac{e_{11}-e_{21}}{C_{11}}\frac{1}{r}\psi'_{mn}-\frac{n^2e_{24}}{C_{11}}\frac{1}{r^2}\psi_{mn}\right\}e^{in\theta}=\frac{\alpha_r}{C_{11}}T_{,r}+\left(\frac{\alpha_r-\alpha_\theta}{C_{11}}\right)\frac{1}{r}T\\
&\sum_{n=-\infty}^{n=+\infty}\sum_{m=1}^{\infty}\left\{v''_{mn}+\frac{1}{r}v'_{mn}-\left(1+\frac{n^2C_{22}}{C_{44}}\right)\frac{1}{r^2}v_{mn}+in\left(1+\frac{C_{12}}{C_{44}}\right)\frac{1}{r}u'_{mn}+in\left(1+\frac{C_{22}}{C_{44}}\right)\frac{1}{r^2}u_{mn}\right.\\
&\left.+in\left(\frac{e_{24}+e_{21}}{C_{44}}\right)\frac{1}{r}\psi'_{mn}+in\frac{e_{24}}{C_{44}}\frac{1}{r^2}\psi_{mn}\right\}e^{in\theta}=\frac{\alpha_\theta}{C_{44}}\frac{1}{r}T_{,\theta}\\
&\sum_{n=-\infty}^{n=+\infty}\sum_{m=1}^{\infty}\left\{u''_{mn}+\left(1+\frac{e_{21}}{e_{11}}\right)\frac{1}{r}u'_{mn}-\frac{n^2e_{24}}{e_{11}}\frac{1}{r^2}u_{mn}+\frac{in(e_{21}+e_{24})}{e_{11}}\frac{1}{r}v'_{mn}-in\frac{e_{24}}{e_{11}}\frac{1}{r^2}v_{mn}-\frac{\eta_{11}}{e_{11}}\psi''_{mn}\right.\\
&\left.-\frac{\eta_{11}}{e_{11}}\frac{1}{r}\psi'_{mn}+\frac{n^2\eta_{22}}{e_{11}}\frac{1}{r^2}\psi_{mn}\right\}e^{in\theta}=-\frac{P_r}{e_{11}}T_{,r}-\frac{P_r}{e_{11}}\frac{1}{r}T-\frac{P_\theta}{e_{11}}\frac{1}{r}T_{,\theta}
\end{aligned}\tag{25}$$

Eqs. (25) are a system of ordinary differential equations with non-constant coefficients having general and particular solutions. The general solutions for $n\neq 0$ are assumed as

$$u_n^g(r)=Rr^\eta\qquad v_n^g(r)=Sr^\eta\qquad \psi_n^g(r)=Wr^\eta\tag{26}$$

Where R,S,W are the unknown constants and by using the specified boundary conditions are determined. Substituting Eqs. (26) into Eqs. (25) yield

$$\left[\eta^2-\frac{C_{22}+n^2C_{44}}{C_{11}}\right]R+i\left[\frac{C_{12}-C_{22}}{C_{11}}\eta\right]nS+\left[\frac{e_{11}}{C_{11}}\eta^2-\frac{e_{21}}{C_{11}}\eta-\frac{n^2e_{24}}{C_{11}}\right]W=0$$
$$i\left[\left(1+\frac{C_{12}}{C_{44}}\right)\eta+\left(1+\frac{C_{22}}{C_{44}}\right)\right]nR+\left[\eta^2-\frac{C_{44}+n^2C_{22}}{C_{44}}\right]S+i\left[\left(\frac{e_{24}+e_{21}}{C_{44}}\right)\eta+\frac{e_{24}}{C_{44}}\right]nW=0 \qquad (27)$$
$$\left[\eta^2+\frac{e_{21}}{e_{11}}\eta-\frac{n^2e_{24}}{e_{11}}\right]R+i\left[\frac{\eta(e_{21}+e_{24})}{e_{11}}-\frac{e_{24}}{e_{11}}\right]nS+\left[\frac{n^2\eta_{22}}{e_{11}}-\frac{\eta_{11}}{e_{11}}\eta^2\right]W=0$$

Eqs. (27) are a system of algebraic equations. For obtaining the nontrivial solution of the equations, the determinant of system should be equal to zero. So the six roots η_{n1} to η_{n6} for the equations are achieved and the general solution is

$$u_n^g(r)=\sum_{k=1}^{6}R_{nk}r^{\eta_{nk}},\quad v_n^g(r)=\sum_{k=1}^{6}M_{nk}R_{nk}r^{\eta_{nk}},\quad \psi_n^g(r)=\sum_{k=1}^{6}N_{nk}R_{nk}r^{\eta_{nk}} \qquad (28)$$

Where M_{nk} is the relation between constants R_{nk} and S_{nk} and N_{nk} is the relation between constants R_{nk} and W_{nk} respectively and are obtained from Eq. (38) as

$$M_{nk}=i\,\frac{a_{nk}c'_{nk}-a'_{nk}c'_{nk}}{b'_{nk}c_{nk}+b_{nk}c'_{nk}}\qquad N_{nk}=-\frac{a_{nk}}{c_{nk}}+\frac{a_{nk}b_{nk}c'_{nk}-a'_{nk}b_{nk}c_{nk}}{b'_{nk}c_{nk}^2+b_{nk}c_{nk}c'_{nk}}\qquad k=1,\dots,6\qquad n\neq 0 \qquad (29)$$

Where a_{nk}, b_{nk}, c_{nk} and a'_{nk}, b'_{nk}, c'_{nk} are given in the appendix. For $n=0$ Eqs. (25) are independent of each other, which yield

$$\sum_{m=1}^{\infty}\left\{u''_{m0}+\frac{1}{r}u'_{m0}-\frac{C_{22}}{C_{11}}\frac{1}{r^2}u_{m0}+\frac{e_{11}}{C_{11}}\psi''_{m0}+\frac{e_{11}-e_{21}}{C_{11}}\frac{1}{r}\psi'_{m0}\right\}=\frac{\alpha_r}{C_{11}}T_{,r}+\left(\frac{\alpha_r-\alpha_\theta}{C_{11}}\right)\frac{1}{r}T \qquad (30)$$

$$\sum_{m=1}^{\infty}\left\{v''_{m0}+\frac{1}{r}v'_{m0}-\frac{1}{r^2}v_{m0}\right\}=\frac{\alpha_\theta}{C_{44}}\frac{1}{r}T_{,\theta} \qquad (31)$$

$$\sum_{m=1}^{\infty}\left\{u''_{m0}+\left(1+\frac{e_{21}}{e_{11}}\right)\frac{1}{r}u'_{m0}-\frac{\eta_{11}}{e_{11}}\psi''_{m0}-\frac{\eta_{11}}{e_{11}}\frac{1}{r}\psi'_{m0}\right\}=-\frac{P_r}{e_{11}}T_{,r}-\frac{P_r}{e_{11}}\frac{1}{r}T-\frac{P_\theta}{e_{11}}\frac{1}{r}T_{,\theta} \qquad (32)$$

Two equations (30) and (32) are a system of ordinary differential equations. The general solution of this case is considered as

$$u_{m0}^g(r,t)=\sum_{k=1}^{4}R_{0k}r^{\eta_{0k}}\qquad v_{m0}^g(r,t)=\sum_{k=5}^{6}R_{0k}r^{\eta_{0k}}\qquad \psi_{m0}^g(r,t)=\sum_{k=1}^{4}Z_{0k}R_{0k}r^{\eta_{0k}} \qquad (33)$$

Where

$$\eta_{01,2}=\pm\sqrt{\frac{\eta_{11}C_{22}+e_{21}^2}{\eta_{11}C_{11}+e_{11}^2}},\quad \eta_{03,4}=0,\quad \eta_{05,6}=\pm 1,\quad Z_{0k}=\frac{e_{11}\eta_{0k}^2+e_{21}\eta_{0k}}{\eta_{11}\eta_{0k}^2}\qquad k=1,\dots,4 \qquad (34)$$

The particular solutions u_n^p, v_n^p and ψ_n^p of Eqs. (25) for $n\neq 0$ are assumed as

$$u_{mn}^{p}(r,t)=r\sum_{k=0}^{\infty}\left[D_{mnk1}J_n(\lambda_{mn}r)+D_{mnk2}Y_n(\lambda_{mn}r)\right]G_{mn}(t)+D_{mn3}r^2+D_{mnk4}r^3$$
$$v_{mn}^{p}(r,t)=r\sum_{k=0}^{\infty}\left[D_{mnk5}J_n(\lambda_{mn}r)+D_{mnk6}Y_n(\lambda_{mn}r)\right]G_{mn}(t)+D_{mn7}r^2+D_{mn8}r^3 \quad (35)$$
$$\psi_{mn}^{p}(r,t)=r\sum_{k=0}^{\infty}\left[D_{mnk9}J_n(\lambda_{mn}r)+D_{mnk10}Y_n(\lambda_{mn}r)\right]G_{mn}(t)+D_{mn11}r^2+D_{mn12}r^3$$

we consider the definition of Bessel function of the first and second type as

$$J_n(\lambda_{mn}r)=\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{1}{2}\lambda_{mn}r\right)^{2k+n}}{k!\,\Gamma(k+n+1)} \qquad Y_n(\lambda_{mn}r)=\frac{J_n(\lambda_{mn}r)\cos n\pi-(-1)^n J_n(\lambda_{mn}r)}{\sin n\pi} \quad (36)$$

Substituting Eqs. (35) and (36) and using heat distribution in piezoelectric layers into Eqs. (25) yield

$$\begin{cases}D_{mnk1}x_1+D_{mnk5}x_2+D_{mnk9}x_3=x_4\\ D_{mnk1}x_{11}+D_{mnk5}x_{12}+D_{mnk9}x_{13}=x_{14}\\ D_{mnk1}x_{21}+D_{mnk5}x_{22}+D_{mnk9}x_{23}=x_{24}\end{cases} \qquad \begin{cases}D_{mnk2}x_1+D_{mnk6}x_2+D_{mnk10}x_3=c_{mn}x_4\\ D_{mnk2}x_{11}+D_{mnk6}x_{12}+D_{mnk10}x_{13}=c_{mn}x_{14}\\ D_{mnk2}x_{21}+D_{mnk6}x_{22}+D_{mnk10}x_{23}=c_{mn}x_{24}\end{cases} \quad (37)$$

$$\begin{cases}D_{mn3}x_5+D_{mn7}x_6+D_{mn11}x_7=0\\ D_{mn3}x_{15}+D_{mn7}x_{16}+D_{mn11}x_{17}=0\\ D_{mn3}x_{25}+D_{mn7}x_{26}+D_{mn11}x_{27}=0\end{cases} \qquad \begin{cases}D_{mn4}x_8+D_{mn8}x_9+D_{mn12}x_{10}=0\\ D_{mn4}x_{18}+D_{mn8}x_{19}+D_{mn12}x_{20}=0\\ D_{mn4}x_{28}+D_{mn8}x_{29}+D_{mn12}x_{30}=0\end{cases} \quad (38)$$

Eqs. (37) and (38) are four systems of algebraic equations. The determinant of coefficients Eqs. (38) are zero, so the obvious answer is the only possible. x_1 to x_{30} and D_{mnk1} to D_{mn12} are given in the Appendix. The particular solutions of Eqs. (30)-(32) for $n=0$ are $u_{m0}^{p}(r,t)$, $v_{m0}^{p}(r,t)$, $\psi_{m0}^{p}(r,t)$ are given in the appendix and substituting these equations into Eqs. (30) - (32) lead to algebraic equation systems. x_{01} to x_{030} are x_1 to x_{30} for $n=0$ and x_{031} to x_{036} and D_{m0k1} to D_{m012} are given in the appendix. The complete solutions $u(r,\theta,t)$, $v(r,\theta,t)$ and $\psi(r,\theta,t)$ are the sum of the general and particular solutions and are

$$u(r,\theta,t)=\sum_{n=-\infty}^{n=+\infty}\left\{\sum_{\substack{k=1\\ ,n\neq0}}^{6}R_{nk}r^{\eta_{nk}}+\sum_{k=1}^{4}R_{0k}r^{\eta_{0k}}+\sum_{m=1}^{\infty}\left\langle r\sum_{k=0}^{\infty}\left[D_{mnk1}J_n(\lambda_{mn}r)+D_{mnk2}Y_n(\lambda_{mn}r)\right]G_{mn}(t)\right.\right.$$
$$\left.\left.+D_{m03}r^2+D_{m04}r^3\right\rangle\right\}e^{in\theta}$$

$$v(r,\theta,t)=\sum_{n=-\infty}^{n=+\infty}\left\{\sum_{\substack{k=1\\ ,n\neq0}}^{6}M_{nk}R_{nk}r^{\eta_{nk}}+\sum_{k=5}^{6}R_{0k}r^{\eta_{0k}}+\sum_{\substack{m=1\\ ,n\neq0}}^{\infty}\left\langle r\sum_{k=0}^{\infty}\left[D_{mnk5}J_n(\lambda_{mn}r)+D_{mnk6}Y_n(\lambda_{mn}r)\right]\right.\right.$$
$$\left.\left.\times G_{mn}(t)+D_{m07}r^2+D_{m08}r^3\right\rangle\right\}e^{in\theta}$$

$$\psi\left(r,\theta,t\right)=\sum_{n=-\infty}^{n=+\infty}\left\{\sum_{\substack{k=1\\,n\neq 0}}^{6}N_{nk}R_{nk}r^{\eta_{nk}}+\sum_{k=1}^{4}Z_{0k}R_{0k}r^{\eta_{0k}}+\sum_{m=1}^{\infty}\left\langle r\sum_{k=0}^{\infty}\left[D_{mnk9}J_{n}(\lambda_{mn}r)+D_{mnk10}Y_{n}(\lambda_{mn}r)\right]\right.\right.$$
$$\left.\left.\times G_{mn}(t)+D_{m011}r^{2}+D_{m012}r^{3}\right\rangle\right\}e^{in\theta} \qquad (39)$$

Substituting Eqs. (39) into Eq. (1), (2) the stress are obtained as

$$\sigma_{1rr}=\sum_{n=-\infty}^{n=+\infty}\left\{\sum_{\substack{k=1\\,n\neq 0}}^{6}\left(C_{11}\eta_{nk}+C_{12}\left(in\,M_{nk}+1\right)+e_{11}N_{nk}\eta_{nk}\right)R_{nk}r^{\eta_{nk}-1}+\sum_{k=1}^{4}\left(C_{11}\eta_{0k}+C_{12}+e_{11}Z_{0k}\eta_{0k}\right)\right.$$
$$\times R_{0k}r^{\eta_{0k}-1}+\sum_{m=1}^{\infty}\left\langle\sum_{k=0}^{\infty}\left[\left(C_{11}\left(2k+n+1\right)+C_{12}\right)D_{mnk1}J_{n}\left(\lambda_{mn}r\right)+\left(C_{11}\left(2k+n+1\right)+C_{12}\right)\right.\right.$$
$$\left.\left.\times D_{mnk2}Y_{n}\left(\lambda_{mn}r\right)\right]G_{mn}(t)+\left(2C_{11}+C_{12}\right)D_{m03}r+\left(3C_{11}+C_{12}\right)D_{m04}r^{2}\right\rangle+\sum_{k=5}^{6}inC_{12}R_{0k}$$
$$\times r^{\eta_{0k}-1}+inC_{12}\sum_{\substack{m=1\\,n\neq 0}}^{\infty}\left\langle\sum_{k=0}^{\infty}\left[D_{mnk5}J_{n}(\lambda_{mn}r)+D_{mnk6}Y_{n}(\lambda_{mn}r)\right]G_{mn}(t)+D_{m07}r+D_{m08}r^{2}\right\rangle$$
$$+e_{11}\sum_{m=1}^{\infty}\left\langle\sum_{k=0}^{\infty}\left[D_{mnk9}\left(2k+n+1\right)J_{n}(\lambda_{mn}r)+D_{mnk10}\left(2k+n+1\right)Y_{n}(\lambda_{mn}r)\right]G_{mn}(t)+2D_{m011}r\right.$$
$$\left.\left.+3D_{m012}r^{2}\right\rangle\right\}e^{in\theta}-\alpha_{r}\sum_{n=-\infty}^{n=+\infty}\sum_{m=1}^{\infty}C_{mn}(\lambda_{mn}r)\left\{e^{-\int\tau dt}\left[b_{n}+\int\frac{R^{*}(t)}{2\pi\|C_{mn}(\lambda_{mn}r)\|^{2}}e^{\int\tau dt}dt\right]\right\}e^{in\theta}$$
$$-\alpha_{r}Ar^{2}-\alpha_{r}Br$$

$$\sigma_{r\theta}=\sum_{n=-\infty}^{n=+\infty}\left\{\sum_{\substack{k=1\\,n\neq 0}}^{6}\left(C_{44}\left(in+M_{nk}\eta_{nk}-M_{nk}\right)+in\,e_{24}N_{nk}\right)R_{nk}r^{\eta_{nk}-1}+\sum_{k=1}^{4}\left(C_{44}+e_{24}Z_{0k}\right)in\,R_{0k}r^{\eta_{0k}-1}\right.$$
$$+inC_{44}\sum_{m=1}^{\infty}\left\langle\sum_{k=0}^{\infty}\left[D_{mnk1}J_{n}\left(\lambda_{j\,mn}r\right)+D_{mnk2}Y_{n}\left(\lambda_{j\,mn}r\right)\right]G_{j\,mn}(t)+D_{m03}r+D_{m04}r^{2}\right\rangle$$
$$+\sum_{k=5}^{6}C_{44}\left(\eta_{0k}-1\right)R_{0k}r^{\eta_{0k}-1}+C_{44}\sum_{\substack{m=1\\,n\neq 0}}^{\infty}\left\langle\sum_{k=0}^{\infty}\left[D_{mnk5}(2k+n)J_{n}(\lambda_{j\,mn}r)+D_{mnk6}\left(2k+n\right)Y_{n}(\lambda_{j\,mn}r)\right]\right.$$
$$\left.\times G_{j\,mn}(t)+D_{m07}r+2D_{m08}r^{2}\right\rangle+in\,e_{24}\sum_{m=1}^{\infty}\left\langle\sum_{k=0}^{\infty}\left[D_{mnk9}J_{n}(\lambda_{mn}r)+D_{mnk10}+Y_{n}(\lambda_{j\,mn}r)\right]G_{j\,mn}(t)\right.$$
$$\left.\left.+D_{m011}r+D_{m012}r^{2}\right\rangle\right\}e^{in\theta}$$

$$\sigma_{\theta\theta} = \sum_{n=-\infty}^{n=+\infty} \left\{ \sum_{\substack{k=1\\,n\neq 0}}^{6} \left(C_{12}\eta_{nk} + C_{22}\left(in M_{nk} + 1 \right) + e_{21} N_{nk}\eta_{nk} \right) R_{nk}\, r^{\eta_{nk}-1} + \sum_{k=1}^{4} \left(C_{12}\eta_{0k} + C_{22} + e_{21} Z_{0k}\eta_{0k} \right) \right.$$

$$\times R_{0k} r^{\eta_{0k}-1} + \sum_{m=1}^{\infty} \left\langle \sum_{k=0}^{\infty} \left[\left(C_{12}\left(2k+n+1\right) + C_{22} \right) D_{mnk1} J_n\left(\lambda_{mn} r\right) + \left(C_{12}\left(2k+n+1\right) + C_{22} \right) Y_n\left(\lambda_{mn} r\right) \right] \right.$$

$$\times G_{mn}(t) + \left(2C_{12} + C_{22}\right) D_{m03} r + \left(3C_{12} + C_{22}\right) D_{m04} r^2 \Big\rangle + \sum_{k=5}^{6} in C_{22} R_{0k} r^{\eta_{0k}-1} + in C_{22} \sum_{\substack{m=1\\,n\neq 0}}^{\infty} \left\langle \sum_{k=0}^{\infty} \left[D_{mnk5} \right. \right.$$

$$\times J_n(\lambda_{mn} r) + D_{mnk6} Y_n(\lambda_{mn} r) \Big] G_{mn}(t) + D_{m07} r + D_{m08} r^2 \Big\rangle + e_{21} \sum_{m=1}^{\infty} \left\langle \sum_{k=0}^{\infty} \left[D_{mnk9}\left(2k+n+1\right) J_n(\lambda_{mn} r) \right. \right.$$

$$+ D_{mnk10}\left(2k+n+1\right) Y_n(\lambda_{mn} r) \Big] G_{mn}(t) + 2D_{m011} r + 3D_{m012} r^2 \Big\rangle \Bigg\} e^{in\theta} - \alpha_\theta \sum_{n=-\infty}^{n=+\infty} \sum_{m=1}^{\infty} C_{mn}(\lambda_{mn} r)$$

$$\times \left\{ e^{-\int \tau dt} \left[b_n + \int \frac{R^*(t)}{2\pi \| C_{mn}(\lambda_{mn} r) \|^2} e^{\int \tau dt} dt \right] \right\} e^{in\theta} - \alpha_\theta A_j r^2 - \alpha_\theta B_j r \tag{40}$$

It is recalled that $R_{nk} (k = 1, \ldots, 6)$ are six unknown constants and therefore to determine these constants, six boundary conditions that may be either the given displacements or stresses, or combinations are required.

$$\begin{matrix} u(a,\theta) = f_1(\theta) & u(b,\theta) = f_4(\theta) & & \\ v(a,\theta) = f_2(\theta) & v(b,\theta) = f_5(\theta) & \sigma_{rr}(a,\theta) = f_7(\theta) & \sigma_{rr}(b,\theta) = f_9(\theta) \\ \psi(a,\theta) = f_3(\theta) & \psi(b,\theta) = f_6(\theta) & \sigma_{r\theta}(a,\theta) = f_8(\theta) & \sigma_{r\theta}(b,\theta) = f_{10}(\theta) \end{matrix} \tag{41}$$

Expanding these boundary conditions into complex Fourier series and select from the list of Eqs. (41) and using the continuity conditions between layers lead to a system of six linear equations to be solved for the constants $R_{nk} (k = 1, \ldots, 6)$.

4 Numerical results and discussion

The analytical solutions obtained in the previous section are checked. Assume piezoelectric material properties PZT-4 from following table

Table 1 Material properties of piezoelectric

Material	**Elastic constants, Gpa**						
PZT-4	C_{11}	C_{12}	C_{13}	C_{22}	C_{23}	C_{33}	C_{44}
	139	78	74	139	74	115	5.6

Piezoelectric constants, C/m^2 — Permittivity, $10^{-9} C/Nm^2$
Pyroelectric constants, $10^{-5} C/Km^2$ — Coefficient of thermal expansion, $1/K$

PZT-4	e_{11}	e_{12}	e_{22}	η_{11}	η_{22}	$P_r = P_\theta = P_0$	$\alpha_r = \alpha_\theta = \alpha_0$
	-5.2	15.1	12.7	6.5	6.5	5.4	2.62

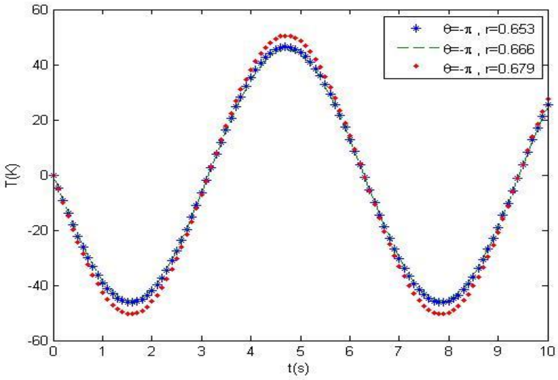

Figure 1 Transient temperature distribution in hollow cylinde

Let us consider a thick hollow cylinder of radii $a = 0.64\, m$ and $b = 0.68\, m$ and thermal conductivity, density and specific heat capacity are $k = 1.5\, W/m\, K$, $\rho = 7500\, kg/m^3$ and $c = 350\ J/kg\, K$ respectively. As the example, Consider a hollow cylinder where the inside boundary is fixed with zero temperature and the outside boundary is assumed to be traction-free with given temperature distribution $g_2(\theta, t) = 20 \sin(t) \cos(2\theta)\ K.$
Therefore, the assumed boundary conditions result in $u(a, \theta) = 0$, $v(a, \theta) = 0$, $\sigma_{rr}(b, \theta) = 0$, $\sigma_{r\theta}(b, \theta) = 0$ and $g_1(\theta, t) = 0$. In this example the electrical boundary conditions are $\psi(a, \theta) = 0$ and $\psi(b, \theta) = 30 \cos\ (2\theta)\ v$ and. The initial temperature is zero. The cylinder is heated by the rate of energy generation per unit time and unit volume of $(r, \theta, t) = 6 \times 10^6 \times \frac{1}{r} \sin(t) \cos(2\theta) \frac{W}{m^3}$.

r

Figure. (1) illustrates the cylinder temperature at $\theta = -\pi$ over the course of 10 seconds. The temperature on the vertical axis is plotted against the time in seconds on the horizontal axis. The results are the sum of transient and steady state solutions that depend upon the initial condition for temperature and heat source, respectively. With small range of radius, the temperature distribution in the piezoelectric layers is close, but as may be seen temperature increases as radius increase.
Figure. (2) shows temperature distribution for different radius r and time t at $\theta = -\pi$. Radial and shear stresses distribution along the thickness and for a specific angle over the course of 10 seconds are show in the Figures. (3),(4), can easily be seen from these figures which radial and circumferential stresses decreases when the radius increases. To verify the proposed method, consider the piezoelectric cylinder with similar boundary conditions and inner radius $\mathrm{a} = 0.64\ \mathrm{m}$ and outer radius and $\mathrm{b} = 0.68\ \mathrm{m}$. Figure (5) illustrates radial stress in piezoelectric cylinder along the thickness by present and finite element method. The results are in good agreement with obtained results from finite element method.

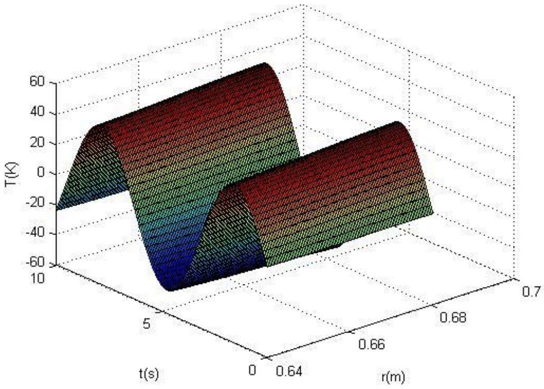

Figure 2 temperature distribution at $\theta = -\pi$

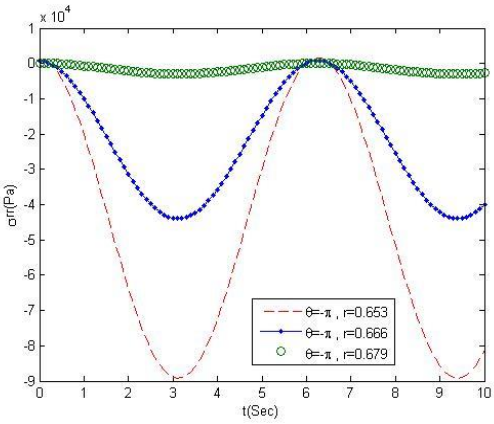

Figure 3 Radial stress

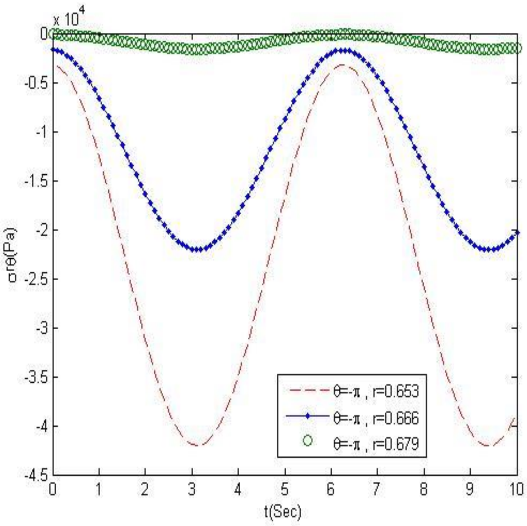

Figure 4 Hoop stress

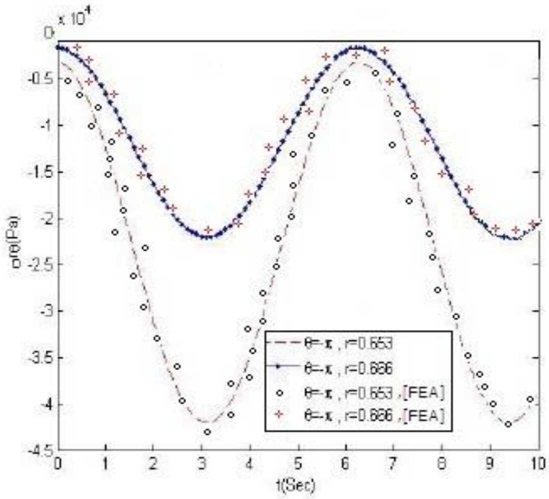

Figure 5 The comparison of Hoop stress along the thickness by present and the finite element method

5 Conclusions

This paper presents a direct method of solution to obtain the transient mechanical and thermal stresses in a piezoelectric cylinder with heat source. The advantage of this method, compared to the conventional potential function method, is its mathematical strength to handle more general types of the mechanical and thermal boundary conditions. More complicated mechanical and thermal boundary conditions may be handled using the proposed method.

The curves associated with the non-zero heat source follow the sine-form pattern of the assumed heat source. Temperature distribution are zero at t=0 due to the initial temperature. According to the given mechanical boundary conditions, stresses at the outside and displacements at the inside surface are zero and temperature increases as radius increase.

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Nomenclature

a, b	inner and outer radius of the hollow cylinder
T_i	temperature distribution
k_i	thermal conductivity
ρ_i	mass density
c_i	specific heat capacity
R	energy source
r	radial coordinate

θ	circumferential coordinate
t	time
$X_{ij}\left(i,j=1,2\right)$	Robine-type constants
$g_1\left(\theta,t\right)$	ambient temperature inner the cylinder
$g_2\left(\theta,t\right)$	ambient temperature outer the cylinder
$g_3\left(r,t\right)$	initial temperature for the cylinder
λ_{mn}	eigenvalues for the cylinder
J_n, Y_n	Bessel functions for the first and the second kinds of order n
u, v	Displacement components
C	Elastic coefficient
e	Piezoelectric coefficient
η	Dielectric constant
P	Pyroelectric constant
α	Thermal modulus
υ	Poisson's ratio
ε_{ij}	Strain tensor $(i,j)=(r,\theta)$
σ_{ij}	Stress tensor $(i,j)=(r,\theta)$

Appendix

$$a_{nk}=\eta_{nk}^{2}-\frac{C_{22}+n^{2}C_{44}}{C_{11}},\qquad b_{nk}=\left[\frac{C_{12}-C_{22}}{C_{11}}\eta_{nk}\right]n$$

$$c_{nk}=\frac{e_{11}}{C_{11}}\eta_{nk}^{2}-\frac{e_{21}}{C_{11}}\eta_{nk}-\frac{n^{2}e_{24}}{C_{11}},\qquad a'_{nk}=\left[(1+\frac{C_{12}}{C_{44}})\eta_{nk}+(1+\frac{C_{22}}{C_{44}})\right]n,\qquad b'_{nk}=\eta_{nk}^{2}-\frac{C_{44}+n^{2}C_{22}}{C_{44}}$$

$$c'_{nk}=\left[(\frac{e_{24}+e_{21}}{C_{44}})\eta_{nk}+\frac{e_{24}}{C_{44}}\right]n,\qquad D_{mn3}=D_{mn4}=D_{mn7}=D_{mn8}=D_{mn11}=D_{mn12}=0$$

$$x_{1}=(2k+n)(2k+n+1)+3(2k+n)+\left(1-\frac{C_{22}+n^{2}C_{44}}{C_{11}}\right),\quad x_{2}=\frac{in(C_{12}+C_{44})}{C_{11}}(2k+n)+\frac{in(C_{12}-C_{22})}{C_{11}}$$

$$x_{3}=\frac{e_{11}}{C_{11}}(2k+n)(2k+n+1)+\left(\frac{3e_{11}-e_{21}}{C_{11}}\right)(2k+n)+\left(\frac{e_{11}-e_{21}-n^{2}e_{24}}{C_{11}}\right),\quad x_{4}=\frac{\alpha_{r}}{C_{11}}(2k+n)+\frac{(\alpha_{r}-\alpha_{\theta})}{C_{11}}$$

$$x_{5}=4-\frac{C_{22}+n^{2}C_{44}}{C_{11}},\qquad x_{6}=\frac{in(2C_{12}+C_{44}-C_{22})}{C_{11}},\qquad x_{7}=\frac{4e_{11}-2e_{21}-n^{2}e_{24}}{C_{11}}$$

$$x_{8}=9-\frac{C_{22}+n^{2}C_{44}}{C_{11}},\qquad x_{9}=\frac{in(3C_{12}+2C_{44}-C_{22})}{C_{11}},\qquad x_{10}=\frac{9e_{11}-3e_{21}-n^{2}e_{24}}{C_{11}}$$

$$x_{11}=in\left(1+\frac{C_{12}}{C_{44}}\right)(2k+n)+in\left(\frac{2C_{44}+C_{12}+C_{22}}{C_{44}}\right)$$

$$x_{12}=(2k+n)(2k+n+1)+3(2k+n)-\left(\frac{n^{2}C_{22}}{C_{44}}\right)$$

$$x_{13}=in\left(\frac{e_{24}+e_{21}}{C_{44}}\right)(2k+n)+in\left(\frac{2e_{24}+e_{21}}{C_{44}}\right),\quad x_{14}=\frac{in\,\alpha_{\theta}}{C_{44}},\quad x_{15}=in\left(\frac{3C_{44}+2C_{12}+C_{22}}{C_{44}}\right)$$

$$x_{16}=3-\frac{n^{2}C_{22}}{C_{44}}\qquad x_{17}=in\left(\frac{3e_{24}+2e_{21}}{C_{44}}\right),\quad x_{18}=in\left(\frac{4C_{44}+3C_{12}+C_{22}}{C_{44}}\right),\ x_{19}=8-\frac{n^{2}C_{22}}{C_{44}}$$

$$x_{20}=in\left(\frac{4e_{24}+3e_{21}}{C_{44}}\right),\quad x_{21}=(2k+n)(2k+n-1)+\left(3+\frac{e_{21}}{e_{11}}\right)(2k+n)+\left(\frac{e_{11}+e_{21}-n^{2}e_{24}}{e_{11}}\right)$$

$$x_{22}=in\frac{(e_{21}+e_{24})}{e_{11}}(2k+n)+\left(in\frac{e_{21}}{e_{11}}\right)$$

$$x_{23}=-\left(\frac{\eta_{11}}{e_{11}}\right)(2k+n)(2k+n-1)-3\left(\frac{\eta_{11}}{e_{11}}\right)(2k+n)+\left(\frac{n^{2}\eta_{22}-\eta_{11}}{e_{11}}\right)$$

$$x_{24}=-\frac{P_{r}}{e_{11}}(2k+n)-\frac{P_{r}}{e_{11}}-in\frac{P_{\theta}}{e_{11}},\qquad x_{25}=\frac{4e_{11}+2e_{21}-n^{2}e_{24}}{e_{11}},\qquad x_{26}=in\left(\frac{2e_{21}+e_{24}}{e_{11}}\right)$$

$$x_{27}=\frac{n^{2}\eta_{22}-4\eta_{11}}{e_{11}},\quad x_{28}=\frac{9e_{11}+3e_{21}-n^{2}e_{24}}{e_{11}},\quad x_{29}=in\left(\frac{3e_{21}+2e_{24}}{e_{11}}\right),\quad x_{30}=\frac{n^{2}\eta_{22}-9\eta_{11}}{e_{11}}$$

$$x_{031}=B_{1}\left(\frac{3\alpha_{r}-\alpha_{\theta}}{C_{11}}\right),\quad x_{032}=A_{1}\left(\frac{3\alpha_{r}-\alpha_{\theta}}{C_{11}}\right),\quad x_{033}=B_{1,\theta}\left(\frac{\alpha_{\theta}}{C_{44}}\right),\qquad x_{034}=A_{1,\theta}\left(\frac{\alpha_{\theta}}{C_{44}}\right)$$

$$x_{035}=-\frac{2P_{r}}{e_{11}}B_{1}-\frac{P_{r}}{e_{11}}B_{1,\theta},\quad x_{036}=-\frac{3P_{r}}{e_{11}}A_{1}-\frac{P_{r}}{e_{11}}A_{1,\theta}$$

$$D_{mnk1}=\frac{\begin{vmatrix}x_4 & x_2 & x_3\\ x_{14} & x_{12} & x_{13}\\ x_{24} & x_{22} & x_{23}\end{vmatrix}}{\begin{vmatrix}x_1 & x_2 & x_3\\ x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\end{vmatrix}},\quad D_{mnk5}=\frac{\begin{vmatrix}x_1 & x_4 & x_3\\ x_{11} & x_{14} & x_{13}\\ x_{21} & x_{24} & x_{23}\end{vmatrix}}{\begin{vmatrix}x_1 & x_2 & x_3\\ x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\end{vmatrix}},\quad D_{mnk9}=\frac{\begin{vmatrix}x_1 & x_2 & x_4\\ x_{11} & x_{12} & x_{14}\\ x_{21} & x_{22} & x_{24}\end{vmatrix}}{\begin{vmatrix}x_1 & x_2 & x_3\\ x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\end{vmatrix}}$$

$$D_{mnk2}=\frac{\begin{vmatrix}c_{mn}x_4 & x_2 & x_3\\ c_{mn}x_{14} & x_{12} & x_{13}\\ c_{mn}x_{24} & x_{22} & x_{23}\end{vmatrix}}{\begin{vmatrix}x_1 & x_2 & x_3\\ x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\end{vmatrix}},D_{mnk6}=\frac{\begin{vmatrix}x_1 & c_{mn}x_4 & x_3\\ x_{11} & c_{mn}x_{14} & x_{13}\\ x_{21} & c_{mn}x_{24} & x_{23}\end{vmatrix}}{\begin{vmatrix}x_1 & x_2 & x_3\\ x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\end{vmatrix}},D_{mnk10}=\frac{\begin{vmatrix}x_1 & x_2 & c_{mn}x_4\\ x_{11} & x_{12} & c_{mn}x_{14}\\ x_{21} & x_{22} & c_{mn}x_{24}\end{vmatrix}}{\begin{vmatrix}x_1 & x_2 & x_3\\ x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\end{vmatrix}}$$

$$u^p_{m0}(r,t)=r\sum_{k=0}^{\infty}\left[D_{m0k1}J_0(\lambda_{m0}r)+D_{m0k2}Y_0(\lambda_{m0}r)\right]G_{m0}(t)+D_{m03}r^2+D_{m04}r^3$$

$$v^p_{m0}(r,t)=r\sum_{k=0}^{\infty}\left[D_{m0k5}J_0(\lambda_{m0}r)+D_{m0k6}Y_0(\lambda_{m0}r)\right]G_{m0}(t)+D_{m07}r^2+D_{m08}r^3$$

$$\psi^p_{m0}(r,t)=r\sum_{k=0}^{\infty}\left[D_{m0k9}J_0(\lambda_{m0}r)+D_{m0k10}Y_0(\lambda_{m0}r)\right]G_{m0}(t)+D_{m011}r^2+D_{m012}r^3$$

$$D_{m0k1}=\frac{\begin{vmatrix}x_{04} & x_{03}\\ x_{024} & x_{023}\end{vmatrix}}{\begin{vmatrix}x_{01} & x_{03}\\ x_{021} & x_{023}\end{vmatrix}},\qquad D_{m0k2}=\frac{\begin{vmatrix}c_{mn}x_{04} & x_{03}\\ c_{mn}x_{024} & x_{023}\end{vmatrix}}{\begin{vmatrix}x_{01} & x_{03}\\ x_{021} & x_{023}\end{vmatrix}}\qquad D_{m03}=\frac{\begin{vmatrix}x_{031} & x_{07}\\ x_{035} & x_{027}\end{vmatrix}}{\begin{vmatrix}x_{05} & x_{07}\\ x_{025} & x_{027}\end{vmatrix}},$$

$$D_{m04}=\frac{\begin{vmatrix}x_{032} & x_{010}\\ x_{036} & x_{030}\end{vmatrix}}{\begin{vmatrix}x_{08} & x_{010}\\ x_{028} & x_{030}\end{vmatrix}},\qquad D_{m0k9}=\frac{\begin{vmatrix}x_{01} & x_{04}\\ x_{021} & x_{024}\end{vmatrix}}{\begin{vmatrix}x_{01} & x_{03}\\ x_{021} & x_{023}\end{vmatrix}}\qquad D_{m0k10}=\frac{\begin{vmatrix}x_{01} & c_{mn}x_{04}\\ x_{021} & c_{mn}x_{024}\end{vmatrix}}{\begin{vmatrix}x_{01} & x_{03}\\ x_{021} & x_{023}\end{vmatrix}},\qquad ,$$

$$D_{m011}=\frac{\begin{vmatrix}x_{05} & x_{031}\\ x_{025} & x_{035}\end{vmatrix}}{\begin{vmatrix}x_{05} & x_{07}\\ x_{025} & x_{027}\end{vmatrix}},\qquad D_{m012}=\frac{\begin{vmatrix}x_{08} & x_{032}\\ x_{028} & x_{036}\end{vmatrix}}{\begin{vmatrix}x_{08} & x_{010}\\ x_{028} & x_{030}\end{vmatrix}}$$

$$D_{m05}=0\qquad D_{m06}=0\qquad D_{m07}=\frac{x_{033}}{3}\qquad D_{m08}=\frac{x_{034}}{8}$$

چکیده

در این مقاله تحلیل دو بعدی گذرای تنش حرارتی و مکانیکی نامتقارن برای سیلندر توخالی ساخته شده از مواد پیزوالکتریک مورد بررسی قرار گرفته است. در ابتدا توزیع حرارتی گذرا، به عنوان تابعی از جهت‌های شعاعی و محیطی و زمان با شرایط مرزی حرارتی کلی، با استفاده از روش جداسازی متغیرها و تابع عمومی بسل، به دست آمده است که نتایج حاصل از آن مجموع حل‌های حالت گذرا و پایدار است که به ترتیب بستگی به شرایط اولیه دما و منبع حرارت دارند.

در این مقاله شکل کلی شرایط مرزی حرارتی و مکانیکی بر روی سیلندر پیزوالکتریک در نظر گرفته شده است. خواص مواد سیلندر پیزوالکتریک در طول ضخامت یکسان است. در این مقاله برای حل معادلات ناویر روش مستقیم با استفاده از معادله اولر و سری مختلط فوریه به کار گرفته شده است که محدودیت‌های روش تابع پتانسیل را ندارد.