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Effect of Temperature on Free Vibration of Functionally Graded Microbeams

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Modified couple stress theory is applied to study effect of temperature on free vibration of Timoshenko functionally graded microbeams. Due to the interatomic and microstructural reactions of the structures in micro scale, the dynamic behavior of microstructures is predicted more accurately by applying the couple stress theory. In this work, both of the simply supported and clamped boundary conditions are assumed to obtain the natural frequencies of a microbeam structure. Natural frequencies are obtained by minimization of the Lagrange function and applying the Ritz method. Finally, effects of various parameters such as temperature change, power law index, height to length scale parameter ratio, and height to length ratio on natural frequencies of the microbeam are presented and discussed in detail. Results obtained in this work are validated against numerical data given in the literature search.

Keywords: Free vibration, Functionally graded microbeam, Couple stress theory, Temperature

1 Introduction

Advantages of functionally graded materials have raised their usage in various industrial applications including biomedicine, optics, electronics, etc. The wide application of microstructures has led many researchers to study the mechanical behavior of structures made of these materials in micro/nanomechanical systems. According to interatomic and microstructural reactions in these structures, conventional strain based theories of structures may not be used to study the behavior of structures in small scale. Stress couple [1, 2], Non-local [3, 4], strain gradient theories [5, 6], and modified stress couple theory [7] are proposed to study the behavior of structures in this scale.

Many researches has been performed on dynamic behavior study of structures made of functionally graded materials in micro and nano scale. Vibration analysis of microstructure using the nonlocal [8], coupled stress [9] and the strain gradient theories [10] are among these studies. Liang et al. [11] examined the dynamic behavior of FG nano-beam applying the stress-couple theory. Max and Reddy [12] studied the free vibration of Timoshenko's micro-beams

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and obtained their natural frequencies for two different boundary conditions by applying the stress-couple theory. Liang et al. [13] studied the temperature effect on buckling and natural frequency of nanobeams made of homogeneous isotropic materials. Asgari et al. [14] utilized the stress-couple theory to analyze the nonlinear vibration of isotropic beams. Salamat et al. [15] studied the dynamic and static behavior of FG microbeam by modification of third order shear deformation theory with stress-couple theory. Ebrahimi et al. [16] studied the thermal buckling and free vibration analysis of size dependent Timoshenko FG nanobeams in thermal environments. The nonlocal elasticity theory of Eringen is used to obtain the natural frequencies of the nanobeam. Lee et al. [17] analyzed the thermal effect on vibration frequency of a scanning thermal microscope cantilever probe using the Timoshenko beam theory. Soh et al. [18] developed a generalized solution to the coupled thermoelastic vibration of a microscale beam resonator induced by pulsed laser heating. Park and Gao. [19] investigated the static and dynamic problems of size-dependent Euler-Bernoulli microbeams by modified couple stress theory. The dynamic problems of Euler-Bernoulli beams are solved analytically based on the modified couple stress theory by Kong et al.[20]. The free and forced vibration of a laminated functionally graded beam of variable thickness under thermally induced initial stresses is studied by Xiang et al. [21] within the framework of Timoshenko beam theory. Arefi and Zenkour presented an analytical solution for vibration and bending analysis of the three-layerd curved nanobeam [22]. They also studied the vibration and bending analysis of sandwich microbeam with two and three-layer integrated piezo-magnetic face sheets [23, 24]. Using the strain gradient theory, free vibration of functionally graded microbeams is studied by Ansari et al. [25].

According to the literature review, the effect of temperature on free vibration of functionally graded microbeams is not studied previously. This paper, studies the effect of temperature on free vibration of FG microbeams using the Timoshenko assumptions and applying the modified stress-couple theory. The Lagrange function is obtained by the kinetic and potential energies of the beam. Minimizing the Lagrange function using the Ritz method, natural frequencies of the beam are obtained. Simplifying the problem to the homogeneous material, the thermal effects on natural frequencies are compared with results of [13]. Also neglecting the effect of temperature, natural frequencies of FG microbeam are compared with those given by Ref. [25]. The natural frequencies obtained by simplification to homogeneous material and neglecting the temperature effects are well compared with data given in literature search. The results show that the temperature has significant effect on natural frequencies of microbeams made of functionally graded materials.

2 Mathematical formulation

Figure (1) shows a beam of length L and thickness h made of functionally graded materials. Material properties of the beam varies through the thickness and obeys the fraction ratio's law.

$$V_c + V_m = 1 \tag{1}$$

Where, V_c and V_m are volume fractions of ceramic and metal. It is assumed that the bottom of the beam $z = -\frac{h}{2}$ is pure ceramic and the top of the beam $z = \frac{h}{2}$ is pure metal. According to

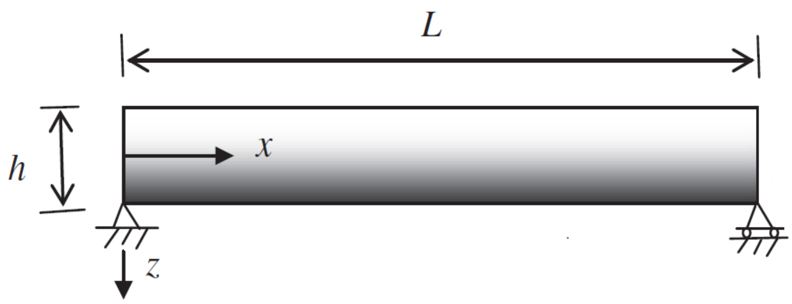

Figure 1 Geometry of the FG-Microbeam

the fraction ratio law, the variation of material properties of the beam through its thickness is

$$E(z) = (E_c - E_m)(\frac{z}{h} + \frac{1}{2})^\beta + E_m \tag{2}$$

$$\alpha(z) = (\alpha_c - \alpha_m)(\frac{z}{h} + \frac{1}{2})^\beta + \alpha_m \tag{3}$$

$$\rho(z) = (\rho_c - \rho_m)(\frac{z}{h} + \frac{1}{2})^\beta + \rho_m \tag{4}$$

where ρ_m and ρ_c are densities, E_m and E_c are modulus of elasticity, and α_m and α_c are thermal expansion coefficient of metal and ceramic respectively. The parameter β is volume fraction power. Applying the stress-couple theory developed by Yang et al. [7], the strain energy U_s in an isotropic linear elastic material which occupies the region L is

$$U_s = \int_V (\sigma_{ij}\epsilon_{ji} + m_{ij}\chi_{ji})dV \tag{5}$$

where σ_{ij} is the Cauchy stress tensor. Definition of infinitesimal strain tensor ϵ_{ij}, symmetric curvature tensor χ_{ij}, and deviatoric part of stress-couple tensor m_{ij} is

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{6}$$

$$\chi_{ij} = \frac{1}{2}(\theta_{i,j} + \theta_{j,i}) \tag{7}$$

$$m_{ij} = 2l^2\mu\chi_{ij} \tag{8}$$

In Eqs. (6) through (8), u_i is the displacement field, μ is the shear modulus and l is the material length scale parameter. The rotation vector θ_i is

$$\theta_i = -\frac{1}{2}\varepsilon_{ijk}u_{j,k} \tag{9}$$

Wherein, ε_{ijk} is permutation symbol.

Based on the Timoshenko beam theory, displacements u_1 and u_3 of the beam along the x and z axis are defined as

$$\begin{aligned} u_1(x,z,t) &= z\phi(x,t) \\ u_3(x,z,t) &= w(x,t) \end{aligned} \tag{10}$$

where ϕ is the transverse rotation about the normal axis y. Substituting the displacement field

(10) into strain tensor (7), the strain-displacement equations are found as follow

$$\begin{aligned}
\epsilon_{xx} &= z\frac{\partial \phi}{\partial x} \\
\epsilon_{xz} &= \frac{1}{2}(\frac{\partial w}{\partial x} + \phi) \\
\epsilon_{yy} &= \epsilon_{zz} = \epsilon_{yz} = \epsilon_{xy} = 0
\end{aligned} \tag{11}$$

According to Eq. (10), components of the rotation vector in Eq. (9) are

$$\begin{aligned}
\theta_y &= \frac{1}{2}(\phi - \frac{\partial w}{\partial x}) \\
\theta_x &= \theta_z = 0
\end{aligned} \tag{12}$$

and components of the symmetric curvature tensor according to the Timoshenko beam theory are

$$\begin{aligned}
\chi_{xy} &= \frac{1}{4}(\frac{\partial \phi}{\partial x} - \frac{\partial^2 w}{\partial x^2}) \\
\chi_{xx} &= \chi_{yy} = \chi_{zz} = \chi_{yz} = \chi_{xz} = 0
\end{aligned} \tag{13}$$

Substituting Eq. (13) into Eq. (8), the only nonzero component of deviatoric stress-couple tensor is

$$m_{xy} = \frac{l^2\mu}{2}\left(\frac{\partial \phi}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right) \tag{14}$$

Applying the generalized elastic stress-strain constitutive law, stress components of the FG nanobeam corresponding to strain components given in Eq. (11) are

$$\sigma_{ij} = 2\mu(z)\,\epsilon_{ij} + [\lambda(z)\epsilon_{kk} - (3\lambda(z) + 2\mu(z))\alpha(z)\Delta T]\delta_{ij} \tag{15}$$

where

$$\lambda = \frac{E(z)(1-\nu)}{(1+\nu)(1-2\nu)}, \mu = \frac{E(z)}{2(1+\nu)} \tag{16}$$

Substituting conditions gieven in Eq. (11) into Eq. (15) results

$$\begin{aligned}
\sigma_{xx} &= \frac{E(z)(1-\nu)}{(1+\nu)(1-2\nu)}\epsilon_{xx} - \frac{E(z)}{(1-2\nu)}\alpha(z)\Delta T \\
\sigma_{xz} &= \frac{E(z)}{(1+\nu)}\epsilon_{xz}
\end{aligned} \tag{17}$$

Upon substitution the stress, strain, symmetric curvature and deviatoric stress-couple tensors into Eq. (5), the strain energy is obtained. Now, the strain energy is divided into thermal and mechanical parts as follows

$$U_S = U_m + U_T \tag{18}$$

where the thermal strain energy $U_T$and mechanical strain energy U_m are

$$U_m = \frac{1}{2}\int_0^L [M_x(\frac{\partial \phi}{\partial x}) + Q_x(\frac{\partial w}{\partial x} + \phi) + \frac{1}{2}Y_{xy}(\frac{\partial \phi}{\partial x} - \frac{\partial^2 w}{\partial x^2})]dx \tag{19}$$

$$U_T = \frac{1}{2}\int_0^L -A_{11}\Delta T(\frac{\partial w}{\partial x})^2 dx \tag{20}$$

In Eqs. (19) and Eq. (20), the bending moment M_x, the shear force Q_x and the torque coupling Y_{xy} are

$$M_x = \int_{-h/2}^{h/2} \sigma_{xx} z dz = D_{11}(\frac{\partial \phi}{\partial x}) \tag{21}$$

$$Y_{xy} = \int_{-h/2}^{h/2} m_{xy} dz = \frac{1}{2} l^2 A_{55}(\frac{\partial \phi}{\partial x} - \frac{\partial^2 w}{\partial x^2}) \tag{22}$$

$$Q_x = \int_{-h/2}^{h/2} \sigma_{xz} dz = A_{55}(\frac{\partial w}{\partial x} + \phi) \tag{23}$$

Constants A_{11}, A_{55} and D_{11} are obtained by calculating the following integrals.

$$A_{55} = \int_{-h/2}^{h/2} \frac{E(z)}{2(1+\nu)} dz \tag{24}$$

$$[A_{11}, D_{11}] = \int_{-h/2}^{h/2} \frac{E(z)(1-\nu)}{(1+\nu)(1-\nu)} \alpha(z)[1, z^2] dz \tag{25}$$

By taking into account the rotational inertia, the kinetic energy of the FG nanobeam is

$$K = \frac{1}{2}\int_\Lambda \rho(z)(\dot{u}_1^2 + \dot{u}_3^2) d\Lambda = \frac{1}{2}\int_0^L [I_1(\dot{u}_1^2) + I_1(\dot{w}^2) + I_2(\dot{\phi}_1^2)] dz \tag{26}$$

Wherein,

$$[I_1, I_2] = \int_{-h/2}^{h/2} \rho(z)[1, z^2] dx \tag{27}$$

The boundary conditions for a beam with both end simply supported are

$$u_1 = w = \frac{\partial \phi}{\partial x} = M_x = 0 \quad \text{at} \quad x = 0 \ \text{ and } \ x = L \tag{28a}$$

and for both end clamped are

$$u_1 = w = \phi = 0 \quad \text{at} \quad x = 0 \ \text{ and } \ x = L \tag{28b}$$

To meet the simply support boundary conditions (28a), u_3 and ϕ are assumed as

$$u_1 = \sum_{n=1}^{N} A_n(t)\sin(\frac{n\pi x}{L})$$

$$u_3 = \sum_{n=1}^{N} B_n(t)\sin(\frac{n\pi x}{L})$$

$$\phi = \sum_{n=1}^{N} C_n(t)\cos(\frac{n\pi x}{L}) \tag{29}$$

and for clamped boundary conditions (30), they are assumed as

$$u_1 = \sum_{n=1}^{N} A_n(t) \sin(\frac{n\pi x}{L})$$
$$u_3 = \sum_{n=1}^{N} B_n(t) \sin(\frac{n\pi x}{L})$$
$$\phi = \sum_{n=1}^{N} C_n(t) \sin(\frac{n\pi x}{L}) \quad (30)$$

The multipliers $A_n(t)$, $B_n(t)$ and $C_n(t)$ in Eqs. (29) and (30) are calculated using the Ritz method. To this aim, the Lagrangian function is defined as

$$\mathscr{L} = K - U_s \quad (31)$$

According to Hamilton's principle, the integral of Lagrangian function should be stationary through the motion of the beam. Applying the Ritz method with the solution given in Eqs. (29) and (30), the functional of Lagrangian (31) is extermized with respect to coefficients $A_n(t)$,$B_n(t)$ and $C_n(t)$as follows

$$\frac{\partial \mathscr{L}}{\partial A_n} = 0, \quad \frac{\partial \mathscr{L}}{\partial B_n} = 0, \quad \frac{\partial \mathscr{L}}{\partial C_n} = 0 \quad (32)$$

Substitution of Eqs. (29) and (30) into Eqs. (19), (20) and (26) and satisfying conditions (32), results

$$([K] - \omega^2[M])\{q\} = 0 \quad (33)$$

In this equation, ω is the natural frequency, $[M]$ is the mass matrix, $[K]$ is the stiffness matrix. Stiffness and mass matrixes are composed of (3×3) sub-matrices and the elements of these sub-matrices are given in appendix. Sub-matrices in [K] and [M] are $(N \times N)$ matrices. Eqs. (33), is s a set of linear and simultaneous algebraic equations. In this equation q is vector of the constants as shown in following equation.

$$\{q\} = \{A_n, B_n, C_n\}^T \quad (34)$$

Solving the system of Eqs. (34), the natural frequencies and the vector of constants are determined.

3 Validation

Results obtained in this paper are compared with results of Ansari et al. [25] for a FG microbeam made of AL/Sic at room temperature and results of Liang et al. [13] for an isotropic microbeam at various temperatures. In Table (1), normalized natural frequencies obtained for various volume fraction powers β are compared between current work and Ref. [25] for the simply supported Timoshenko FG microbeam. In Ref. [25], the strain gradient theory excluding the thermal effect is applied to the microbeam with material properties $E_{Al} = 70\ GPa$, $\nu_{Al} = 0.3$, $\rho_{Al} = 2702\ kg/m^3$ for metal (Aluminium), $E_{Sc} = 427\ GPa$, $\nu_{Sc} = 0.17$, $\rho_{Sc} = 3100\ kg/m^3$ for ceramic (SiC), the thickness to length ratio $h/L = 0.1$ and the length scale parameter $l = 17.6\ \mu m$. In present work the gradient of the poisson's ratio through the thickness of the

Table 1 Comparison of normalized natural frequency $\tilde{\omega}$ between current work and Ref. [25]

	β						
Source	1	0.1	0.6	1.2	2	10	∞
Ref. [25]	0.8538	0.7619	0.6084	0.5470	0.5100	0.4332	0.3863
Present	0.8657	0.8327	0.7286	0.6658	0.61935	0.4873	0.4441

Table 2 Comparison of first three normalized natural frequencies $\tilde{\omega}$ with Ref. [13]

		ΔT					
ω	Source	0	20	40	60	80	100
ω_1	Present	0.3567	0.3415	0.3257	0.3091	0.2915	0.2727
	Ref. [13]	0.3478	0.3322	0.3159	0.2986	0.2804	0.2608
ω_2	Present	1.4098	1.3950	1.3802	1.3651	1.3499	1.3345
	Ref. [13]	1.2890	1.2727	1.2562	1.2394	1.2225	1.2053
ω_3	Present	3.1107	3.0963	3.0818	3.0673	3.0527	3.0381
	Ref. [13]	2.6277	2.6099	2.5920	2.5739	2.5558	2.5374

beam is neglected and the poisson's ratio is assumed to be $\nu = 0.38$. In Table(1), the values of normalized natural frequency of FG-Nanobeam ($\tilde{\omega}$) with various volume fraction powers and double hinged (H-H) boundary conditions are compared with results of Ref. [25]. It is assumed that the ratio of thickness to length is $h/L = 0.1$ and temperature change is $\Delta T = 0^0C$. It can be seen that the results from this study are well compared to those given in Ref. [25]. In normalized natural frequency $\tilde{\omega}$ is defined as $\tilde{\omega} = \omega L\sqrt{I/A}$. Where, I and A are

$$
\begin{aligned}
A &= \int_{-h/2}^{h/2} \frac{E_{AL}(1-\nu)}{(1-2\nu)(1+\nu)} dz \\
I &= \int_{-h/2}^{h/2} \rho_{AL} dz
\end{aligned} \tag{35}
$$

E_{Al} and ρ_{AL} are elastic modulus and density of Aluminium.

In Table (2), first three normalized natural frequencies of a simply supported isotropic microbeam obtained in present work is compared with results of Liang et al. [13] for various values of temperature change. The beam data are $E = 1.44\ GPa$, $\rho = 1220\ kg/m^3$, $\nu = 0.38$, length scale parameter $l = 17.6\ \mu m$, thickness to length scale parameter ratio $h/l = 2$, thickness to length ratio $h/L = 0.1$ and coefficient of thermal expansion $\alpha = 54 \times 10^{-6}/^oC$. Results of data given in Table (2) shows close agreement between data of present work and Ref. [13].

4 Results and discussion

In this paper the temperature effect on natural frequencies of a simply supported FG-micro beam made of Al/Sic materials is studied. The material properties of the beam for all results represented in this section are the same as given for the Table (1) and the length scale parameter is assumed to be $l = 16\mu m$.

Tables (3) and (4) show the effect of temperature, on first natural frequency of a double hinged (H-H) FG microbeam. The thermal expansion coefficient is $\alpha = 54 \times 10^{-6}/^oC$. Data of these tables show that, increasing the temperature decreases the fundamental frequencies of the beam. Also, increasing the volume fraction power decreases natural frequencies. This is due to the fact that the percentage of metal increases with increase in volume fraction power. Increase

Table 3 Fundamental natural frequency of FG-microbeam for $h/l = 1$ and H-H boundary condition

		β						
h/L	ΔT	0	0.1	0.5	1	2	5	10
0.1	0	13.2182	12.7371	11.3982	10.4239	9.3687	8.0740	7.2578
	20	12.9544	12.4811	11.1676	10.2158	9.1886	7.9274	7.1274
	40	12.6850	12.2197	10.9321	10.0034	9.0049	7.7781	6.9946
	60	12.4099	11.9527	10.6915	9.7864	8.8173	7.6259	6.8593
	80	12.1285	11.6795	10.4453	9.5645	8.6257	7.4706	6.7212
	100	11.8404	11.3997	10.1931	9.3373	8.4298	7.3120	6.5802
	120	11.5451	11.1130	9.9346	9.1044	8.2291	7.1498	6.4361
0.15	0	30.1857	29.0875	26.0301	23.8044	21.3938	18.4365	16.5731
	20	29.9247	28.3211	25.8021	23.5986	21.2156	18.2915	16.4441
	40	29.6615	28.5789	25.5719	23.3910	21.0366	18.1454	16.3140
	60	29.3959	28.3211	25.3397	23.1816	20.8547	17.998	16.1830
	80	29.1278	28.061	25.1054	22.9702	20.6719	17.8495	16.0508
	100	28.8573	27.7984	24.8688	22.7569	20.4875	17.6996	15.9175
	120	28.5842	27.5334	24.6300	22.5415	20.3014	17.5486	15.7832

Table 4 Fundamental natural frequency of FG-microbeam for $h/l = 4$ and H-H boundary condition

		β						
h/L	ΔT	0	0.1	0.5	1	2	5	10
0.1	0	1.9400	1.8571	1.6513	1.5299	1.4219	1.2823	1.1612
	20	1.8254	1.7451	1.5497	1.4395	1.3465	1.2238	1.1096
	40	1.7031	1.6254	1.4410	1.3430	1.2665	1.1624	1.0555
	60	1.5713	1.4961	1.3233	1.2391	1.1812	1.0975	0.9984
	80	1.4273	1.3546	1.1941	1.1256	1.0892	1.0286	0.9379
	100	1.2672	1.1964	1.0491	0.9993	0.9887	0.9547	0.8747
	120	1.0836	1.0138	0.8806	0.8545	0.8767	0.8746	0.8032
0.15	0	4.4303	4.2410	3.7711	3.4937	3.2471	2.9280	2.6517
	20	4.3182	4.1316	3.6719	3.4053	3.1732	2.8706	2.6010
	40	4.2032	4.0191	3.5698	3.3146	3.0975	2.8120	2.5493
	60	4.0849	3.9034	3.4648	3.2213	3.0200	2.7522	2.4965
	80	3.9631	3.7842	3.3565	3.1253	2.9404	2.6911	2.4426
	100	3.8374	3.6611	3.2446	3.0262	2.8585	2.6285	2.3875
	120	3.7075	3.5337	3.1286	2.9237	2.7743	2.5644	2.3311

in h/L, increases the fundamental frequencies. On the other hand, comparison of the natural frequencies between Tables (3) and (4) shows that, the length scale parameter has significant effect on fundamental frequencies of the FG microbeam. As it can be seen from these tables, the natural frequencies decrease rapidly with increase in hight to length scale parameter.

Figure (2) compares the fundamental natural frequency of the FG-microbeam between the conventional and stress couple theories. The curves plotted in this figure, correspond to $h/L = 0.1$, $\beta = 5$ and $\Delta T = 60^0 C$. As the figure shows the difference between these two theories increases for small values of height to length scale parameter h/l. The fundamental frequency obtained by stress couple theory is more than twice of the conventional theory for $h/l = 1$, while the difference is negligible for $h/l > 6$. Also, it can be seen that, the predicted natural frequency by modified couple stress theory are higher than those obtained by the classical theory always.

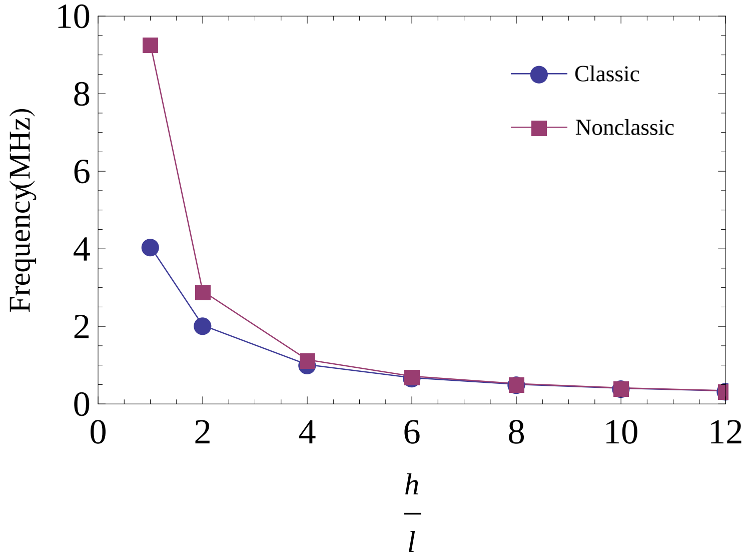

Figure 2 Comparison of fundamental frequency between conventional and stress-couple theories

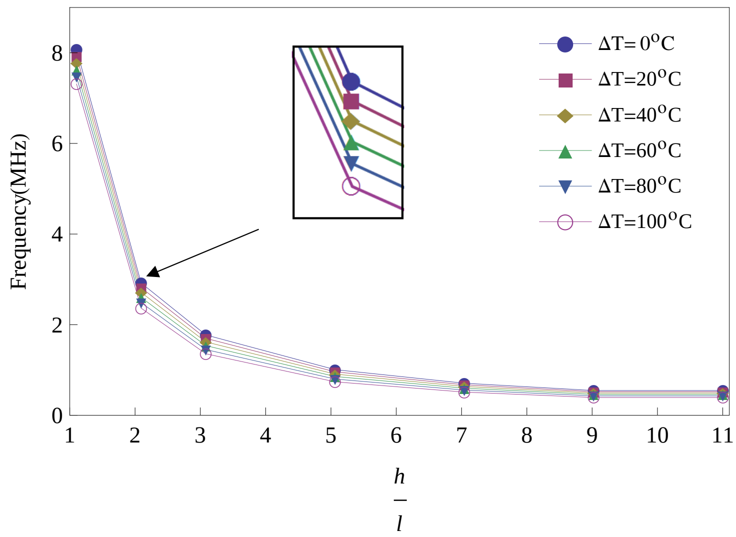

Figure 3 Effect of temperature on variation of natural frequency for $\beta = 5$ and $h/L = 0.1$

The difference between frequencies predicted by these two models is more significant when the thickness of the plate is small, but it is negligible when thickness of the beam is increased. This indicates that the size effect is tangible and must be taken into account when the thickness of the beam is in micron scale. Figure (3) shows fundamental frequencies of the FG microbeam versus the height to length scale parameter ratio at temperature range between 0 to 100^oC. The figure shows that for $h/l < 4$, natural frequencies of the beam are highly affected by the length scale parameter. While, for $h/l > 10$, the natural frequencies are not affected by the length scale parameter. This behavior is seen for all temperature difference values. Table (5) shows the thermal effect on first natural frequency of a C-C FG microbeam. Data of these tables show that, increase in temperature results to decrease in fundamental frequencies of the beam. Also, increasing the volume fraction power decreases natural frequencies. This is due to the fact that the percentage of metal increases with increase in volume fraction power. Increase in h/L, increases the fundamental frequencies. On the other hand, comparison the natural frequencies of Tables (4) and (5) shows that natural frequency values in the same condition for C-C boundary

Table 5 Variation of first natural frequency of FG-microbeam for C-C boundary conditions and $h/l = 4$

		β						
h/L	ΔT	0	0.1	0.5	1	2	5	10
0.1	0	2.2390	2.1710	2.0081	1.9173	1.8408	1.7446	1.6623
	20	2.1405	2.0760	1.9254	1.8460	1.7832	1.7020	1.6266
	40	2.0372	1.9764	1.8390	1.7718	1.7236	1.6584	1.5902
	60	1.9284	1.8716	1.7484	1.6943	1.6619	1.6136	1.5529
	80	1.8130	1.7605	1.6528	1.6132	1.5978	1.5676	1.5147
	100	1.6898	1.6419	1.5513	1.5278	1.5311	1.5201	1.4755
	120	1.4352	1.5141	1.4426	1.4372	1.4612	1.4711	1.4352
0.15	0	8.1821	8.1158	7.9787	7.9228	7.8919	7.8450	7.7838
	20	8.1219	8.0592	7.9323	7.8842	7.8618	7.8238	7.7667
	40	8.0614	8.0021	7.8856	7.8455	7.8316	7.8025	7.7495
	60	8.0003	7.9446	7.8386	7.8065	7.8012	7.7811	7.7323
	80	7.9388	7.8867	7.8013	7.7714	7.7507	7.7397	7.7151
	100	7.8768	7.8284	7.7438	7.7280	7.7402	7.7302	7.6978
	120	7.8143	7.7696	7.6959	7.6885	7.7094	7.7167	7.6805

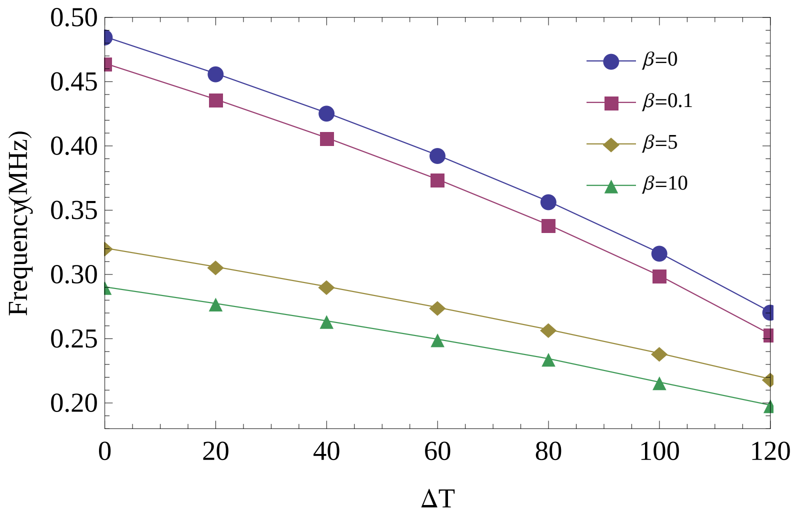

Figure 4 Variation of first natural frequency versus the temperature for $h/L = 0.05$ and $h/l = 1$

conditions is much higher in comparison to H-H boundary conditions. The tables show that, the natural frequencies decrease rapidly with increase in hight to length scale parameter.

Figures (4) and (5) show fundamental frequencies of a H-H FG microbeam versus power law index and temperature. In these figures, the thickness to length scale parameter ratio are $h/l = 1$ and $h/l = 4$ respectively. Data of these Figures show that, increase in temperature results to decrease in fundamental frequencies of the beam. Also, increasing the volume fraction power decreases natural frequencies. Also, increasing the volume fraction power and temperature results to decrease in natural frequencies. This is due to the fact that increase in temperature and volume fraction reduces the stiffness of the beam. Figure (6) compares the plot of first natural frequency versus the volume fraction power between H-H and C-C boundary conditions. In these figures, the thickness to length scale parameter ratios are $h/l = 1$ and $h/L = 0.1$ respectively. As it can be seen the frequencies of the C-C boundary conditions are higher than those of the H-H boundary conditions.

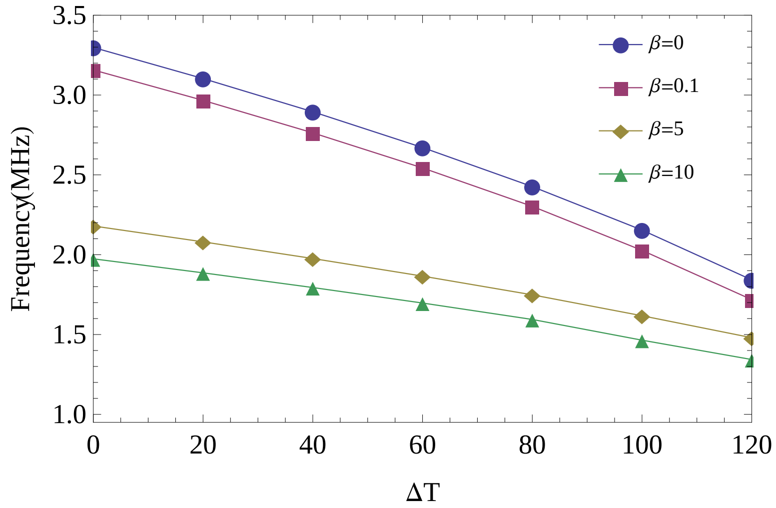

Figure 5 Variation of first natural frequency versus the temperature for $h/L = 0.05$ and $h/l = 4$

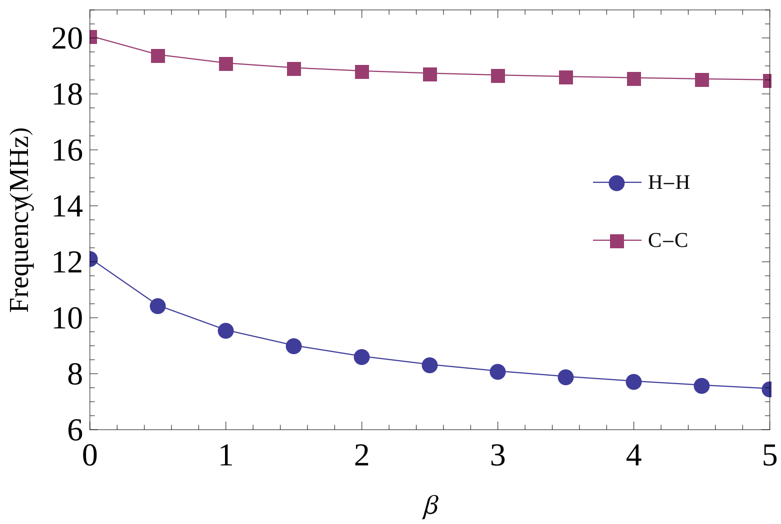

Figure 6 Variation of first natural frequency versus the volume fraction power and boundary conditions for $h/L = 0.1$, $h/l = 1$ and $\triangle T = 80^0 C$

5 Conclusion

In this study, natural frequencies of the FG microbeam are obtained using the stress-couple theory. To this aim, the Lagrangian function of the beam based on the Timoshenko theory is obtained. Applying the Hamilton's principle with the Ritz approximation method, equations of natural frequencies of the beam are solved numerically to investigate effects of the temperature, the length scale parameter, the gradient of material through the thickness and geometric parameters of the beam on natural frequencies. Results obtained using the method applied in this paper are validated with those of given in the literature search for cases of the isotropic microbeam and the FG mircobeam. The numerical results show that, increasing the temperature decreases natural frequencies of the beam. Increasing the volume fraction of metal, also decreases natural frequencies. Comparison of frequencies obtained based on the stress-couple and the conventional theories shows that the difference of these theories is significant for small values of height to length scale ratio.

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Nomenclature

$[K]$	Stiffness matrix
$[M]$	Mass matrix
E_c	Modulus of elasticity of ceramic
E_m	Modulus of elasticity of metal
l	Material length scale parameter

m_{ij}	Stress-couple tensor components
T	Kinetic energy
u_i	Displacement vector components
U_s	Strain energy
V_c	Volume fraction of ceramic
V_m	Volume fraction of metal
q	Vector of constants

Greek Symbols

χ_{ij}	Curvature tensor components
ϵ_{ij}	Strain tensor components
λ	Lame constant
μ	Shear modulus
ρ_c	Density of ceramic
ρ_m	Density of metal
σ_{ij}	Stress tensor components
θ_i	Rotation vector components

Appendix

The stiffness and mass matrix [K], [M] in Eqs. (33) are

$$[K] = \begin{bmatrix} [K_{mn}^{aa}] & [K_{mn}^{ab}] & [K_{mn}^{ac}] \\ & [K_{mn}^{bb}] & [K_{mn}^{bc}] \\ sym & & [K_{mn}^{cc}] \end{bmatrix}, \ [M] = \begin{bmatrix} [M_{mn}^{aa}] & [M_{mn}^{ab}] & [M_{mn}^{ac}] \\ & [M_{mn}^{bb}] & [M_{mn}^{bc}] \\ sym & & [M_{mn}^{cc}] \end{bmatrix} \tag{36}$$

where

$$\begin{aligned} &[K_{mn}^{aa}] = \frac{\partial^2 U_m}{\partial A_m \partial A_n}, \ [K_{mn}^{bb}] = \frac{\partial^2 U_m}{\partial B_m \partial B_n}, \ [K_{mn}^{cc}] = \frac{\partial^2 U_m}{\partial C_m \partial C_n}, \\ &[K_{mn}^{ab}] = \frac{\partial^2 U_m}{\partial A_m \partial B_n}, \ [K_{mn}^{bc}] = \frac{\partial^2 U_m}{\partial B_m \partial C_n}, \ [K_{mn}^{ac}] = \frac{\partial^2 U_m}{\partial A_m \partial C_n}, \\ &[M_{mn}^{aa}] = \frac{\partial^2 w}{\partial A_m \partial A_n}, \ [M_{mn}^{bb}] = \frac{\partial^2 w}{\partial B_m \partial B_n}, \ [M_{mn}^{cc}] = \frac{\partial^2 w}{\partial C_m \partial C_n}, \\ &[M_{mn}^{ab}] = \frac{\partial^2 w}{\partial A_m \partial B_n}, \ [M_{mn}^{bc}] = \frac{\partial^2 w}{\partial B_m \partial C_n}, \ [M_{mn}^{ac}] = \frac{\partial^2 w}{\partial A_m \partial C_n} \end{aligned} \tag{37}$$

In this work a model with 9 degrees of freedom is applied for H-H and C-C boundary conditions. The generalized coordinates of the model are: A_i , B_i and C_i with $i = 1$ to 3. Elements of sub-matrices in the stiffness matrix [K] and the mass matrix [M] for 9 degrees of freedom are

$$[K_{mn}^{aa}] = B \begin{bmatrix} K_{11}^{aa} & K_{12}^{aa} & K_{13}^{aa} \\ & K_{22}^{aa} & K_{23}^{aa} \\ sym & & K_{33}^{aa} \end{bmatrix}, \ [M_{mn}^{aa}] = \begin{bmatrix} M_{11}^{aa} & M_{12}^{aa} & M_{13}^{aa} \\ & M_{22}^{aa} & M_{23}^{aa} \\ sym & & M_{33}^{aa} \end{bmatrix} \tag{38}$$

چکیده

در این مقاله، از تئوری تنش کوپل اصلاح شده برای بررسی ارتعاشات آزاد میکروتیر تیموشنکو ساخته شده از مواد تابعی بهره گرفته شده است. به دلیل در نظر گرفتن اثرات بین اتمی در ریزساختارها در مقیاس میکرو، تئوری تنش کوپل اصلاح شده رفتار دینامیکی میکرو تیر تابعی را با دقت بیشتری پیش‌بینی می‌کند. هر دو شرط مرزی تکیه‌گاه‌های ساده و گیردار برای بدست آوردن فرکانسهای طبیعی تیر در نظر گرفته شده‌اند. با مینیمم کردن تابع لاگرانژین و اعمال روش رتیز، فرکانسهای طبیعی میکروتیر بدست آمده‌اند. نهایتا اثرات پارامترهای مختلف مانند تغییر درجه حرارت، تغییر خواص تابعی ماده، نسبت ارتفاع تیر به پارامتر مقیاس طول و نسبت ارتفاع به طول تیر بر روی فرکانسهای طبیعی میکروتیر نشان داده شده و با جزئیات مورد بررسی قرار گرفته است. به منظور صحت‌سنجی، نتایج عددی بدست آمده در این مقاله با داده‌های موجود در سوابق علمی تحقیق مقایسه شده است.