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Research Paper

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Mojtaba Ayatollahi^{*} Professor Professor Mojtaba Ayatollahi^{*} Professor Plane with Multiple Cracks under Electromechanical Impacts

The analytical method is developed to examine the fracture behavior of a functionally graded piezoelectric rectangular plane (FGPRP) with finite geometry under impact loads. The material properties of the FGPRP vary continuously in the transverse direction. Two different types of boundary conditions are examined and discussed in the analyses. The finite Fourier cosine and Laplace transforms are employed to obtain stress and electric displacement fields in the finite plane containing electro-elastic screw dislocation. Based on the distributed dislocation technique, a set of integral equations for the finite plane is weakened by multiple parallel cracks under electromechanical impact loads. By solving numerically, the resulting singular integral equation, the dynamic stress intensity factor (DSIF) is obtained for the electrically impermeable case. The new results are provided to show the applicability of the proposed solution. The effects of the geometric parameters including plate length, width, crack position, crack length, loading parameter, and FG exponent on the dynamic stress intensity factors are shown graphically and discussed.

Keywords: Transient response, Multiple cracks, Functionally graded material, Piezoelectric rectangular plane, Dynamic stress intensity factors

1 Introduction

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The problems in engineering always contain some cracks and a closed solution cannot be obtained easily. Hence, Green's functions are necessary for these complex problems.

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Furthermore, with the increasing usage of non-homogeneous piezoelectric materials as actuating and sensing devices in smart structures with finite geometry, much attention has been paid to their fracture behavior. To improve the performance of the piezoelectric materials, the functionally graded piezoelectric materials (FGPMs) as a new class of advanced composites have been introduced.

In order to predict the reliable service life of ceramic piezoelectric components, it is necessary to analyze theoretically the damage and fracture processes taking place in piezoelectric materials with consideration of the coupled effects of mechanics and electrics. There are several investigations on fracture analysis of functionally graded piezoelectric materials with infinite or semi-infinite domains under static load. Among a lot of significant efforts in this area, there are limited numbers of investigations dealing with fracture problems in finite domains.

Chang [1] obtained the stress intensity factor of a rectangular orthotropic plate containing a central crack under anti-plane shear by using the Fourier transform and series. The problem of an edge crack in a rectangular sheet subjected to anti-plane shear was examined by Zhang [2]. The problem of an eccentric crack at the interface between two dissimilar layers in a finite rectangular sheet under arbitrary anti-plane shear stress was treated by Zhang and Zhang [3]. The article by Ma [4] dealt with the general solution of the stress intensity factor in a rectangular sheet weakened by a central crack of mode III, where its boundary is constrained. In another paper, Zhang [5] obtained the stress intensity factor of an interface central crack between two orthotropic rectangular sheets. The stress analysis in a nonhomogeneous rectangular sheet with shear modulus varying in the x-direction was accomplished by Zhang and Ban [6]. The dynamic stress intensity factor of a pair of edge cracks in the finite rectangular plate subjected to a normal anti-plane shear wave was analyzed by Zhang [7]. The solution to an eccentric crack problem in a rectangular sheet under anti-plane deformation was the subject of study by Ma and Zhang [8]. Stress intensity factors of an interfacial crack between two dissimilar orthotropic rectangular media were analyzed by Li and Duan [9]. The problem was solved for four types of boundary conditions and the effects of the material properties on stress intensity factors were examined. An orthotropic rectangular plane with various boundary conditions, containing multiple defects was solved by Faal, Daliri, and Milani [10]. In this work, the solution to the anti-plane crack problem was obtained using distributed dislocation technique. They computed the stress intensity factors of crack tips and the dimensionless hoop stresses on the boundary of each cavity. The stress analysis of FGM rectangular planes with several arbitrary smooth cracks was investigated by Faal and Dehgan [11]. In this study, the effects of material properties, crack spacing, and cracks length on the SIF of cracks were investigated. The problem of the cracked rectangular piezoelectric ceramic body under anti-plane mechanical and in-plane electrical loads in the framework of linear piezoelectricity was the subject of an investigation by Kwon and Lee [12]. The paper by Li and Lee [13] was concerned with a crack at an arbitrary position in a rectangular piezoelectric ceramic. The energy release rate was computed and the effect of the crack length on these factors was investigated. An analytical model for a piezoelectric rectangular plane containing multiple cracks and cavities, was treated by Abazadeh and Darafshani [14]. These authors employed the distributed dislocation technique to determine electric displacement, stress intensity factors, and hoop stress around cavities in the piezoelectric rectangular plane. The solution procedures devised in all the above studies are not capable of handling multiple cracks in FGPRP with finite geometry and various boundary conditions under impact electromechanical loads. In general, transient analysis of cracked finite smart structures under dynamic loads is complicated and only those with simple geometries may be handled analytically.

In the present article, we employ the distributed dislocation technique to analyze multiple parallel cracks in an FGPRP with different boundary conditions and subjected to impact loads. By using the finite Fourier cosine and Laplace transforms, the stress and electric displacement

fields in the FGPRP containing screw dislocation were obtained. The dislocation solutions are then used to construct singular integral equations for the FGPRP containing multiple cracks. Two different boundary conditions are examined in the analyses namely, a free-clamped-free clamped case (F-C-F-C) and a clamped-free-free-free case (C-F-F-F). Parametric analyses are carried out to examine the effects of the gradient and loading parameters, crack length, and geometry of the FGPRP on the dynamic stress intensity factors.

2 Formulation of the Problem

We consider a finite rectangular plane made up of functionally graded piezoelectric materials, where the material properties vary continuously in the thickness direction. Under conditions of anti-plane displacement and the in-plane electric fields, the electro-elastic boundary value problem is simplified considerably. The constitutive relations of the non-homogeneous piezoelectric materials are

$$\sigma_{zx}(x,y,t) = c_{44}(y)\frac{\partial w}{\partial x} + e_{15}(y)\frac{\partial \phi}{\partial x}, \quad \sigma_{zy}(x,y,t) = c_{44}(y)\frac{\partial w}{\partial y} + e_{15}(y)\frac{\partial \phi}{\partial y}$$

$$D_{x}(x,y,t) = e_{15}(y)\frac{\partial w}{\partial x} - \varepsilon_{11}(y)\frac{\partial \phi}{\partial x}, \quad D_{y}(x,y,t) = e_{15}(y)\frac{\partial w}{\partial y} - \varepsilon_{11}(y)\frac{\partial \phi}{\partial y}$$
(1)

where c_{44} , e_{15} and ε_{11} are the elastic stiffness, the piezoelectric constant, and the dielectric constant of piezoelectric material, respectively. To overcome the complexity of mathematics, the present work employs exponential functions to describe the continuous variations of material properties,

$$[c_{44}(y), e_{15}(y), \varepsilon_{11}(y), \rho(y)] = [c_{440}, e_{150}, \varepsilon_{110}, \rho_0] e^{2\gamma y}$$
(2)

The governing equations for piezoelectric materials can be expressed as follows

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = \rho(y) \frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0$$
(3)

Substitution of Eqs. (1) and (2) into Eq. (3) yield the governing equations

$$c_{440}\nabla^{2}w + e_{150}\nabla^{2}\phi + 2\gamma c_{440}\frac{\partial w}{\partial y} + 2\gamma e_{150}\frac{\partial \phi}{\partial y} = \rho_{0}\frac{\partial^{2}w}{\partial t^{2}}$$

$$e_{150}\nabla^{2}w - \varepsilon_{110}\nabla^{2}\phi + 2\gamma e_{150}\frac{\partial w}{\partial y} - 2\gamma \varepsilon_{110}\frac{\partial \phi}{\partial y} = 0$$
(4)

where $\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$ is the two-dimensional Laplacian operator in the variables x and y. To solve the present problem, we introduce a new function

$$\varphi^* = \varphi - \alpha W \tag{5}$$

where $\alpha = e_{150} / \varepsilon_{110}$, the constitutive equations can be expressed in terms of new variables as follows:

$$\sigma_{zx}(x,y,t) = ke^{2\gamma y} \frac{\partial w}{\partial x} + e_{150}e^{2\gamma y} \frac{\partial \phi^*}{\partial x}, \quad \sigma_{zy}(x,y,t) = ke^{2\gamma y} \frac{\partial w}{\partial y} + e_{150}e^{2\gamma y} \frac{\partial \phi^*}{\partial y},$$

$$D_x(x,y,t) = -\varepsilon_{110}e^{2\gamma y} \frac{\partial \phi^*}{\partial x}, \quad D_y(x,y,t) = -\varepsilon_{110}e^{2\gamma y} \frac{\partial \phi^*}{\partial y}$$
(6)

The coefficient $k = c_{440} + (e_{150})^2 / \varepsilon_{110}$ is the piezoelectric constant. Under the above consideration, the governing equations can be simplified to the following form:

$$\nabla^{2}w + 2\gamma \frac{\partial w}{\partial y} = \frac{1}{c_{s}^{2}} \frac{\partial^{2}w}{\partial t^{2}}$$

$$\nabla^{2}\phi^{*} + 2\gamma \frac{\partial \phi^{*}}{\partial y} = 0$$
(7)

where $c_s = \sqrt{k / \rho_0}$ denotes the shear wave velocity in the piezoelectric material. To obtain the desired electroelastic field, for convenience it is necessary to impose that the FGP rectangular plane is initially at rest. Namely, the piezoelectric material is subjected to the vanishing initial conditions

$$w(x,y,0) = 0, \quad w_{t}(x,y,0) = 0,$$

 $\phi(x,y,0) = 0, \quad \phi_{t}(x,y,0) = 0,$

(8)

The solution of the governing equations (7) is obtained by using the Fourier cosine, and Laplace transforms technique. The finite Fourier cosine transform f(x, y, t), 0 < x < a is defined as follows:

$$F_{c}(n,y,t) = \int_{0}^{a} f(x,y,t) \cos(\frac{n\pi}{a}x) dx$$
(9)

Moreover, the inverse finite Fourier cosine transform read as:

$$f(x,y,t) = \frac{F_{c}(0,y,t)}{a} + \frac{2}{a} \sum_{n=1}^{\infty} F_{c}(n,y,t) \cos(\frac{n\pi}{a}x)$$
(10)

The Laplace transform of a function with respect to t is defined by

$$\overline{W}(x,y,s) = \int_0^\infty w(x,y,t) e^{-st} dt$$
(11)

where the bar denotes the Laplace transform. Applying finite Fourier cosine and Laplace transforms to Eq. (7) gives

$$\frac{d^{2}\bar{W}_{c}(n,y,s)}{dy^{2}} + 2\gamma \frac{d\bar{W}_{c}(n,y,s)}{dy} - (\lambda_{n})^{2}\bar{W}_{c}(n,y,s) = (-1)^{n+1} \frac{d\bar{w}(a,y,s)}{dx} - \frac{d\bar{w}(0,y,s)}{dx},$$

$$\frac{d^{2}\bar{\Phi}_{c}^{*}(n,y,s)}{dy^{2}} + 2\gamma \frac{d\bar{\Phi}_{c}^{*}(n,y,s)}{dy} - (\beta_{n})^{2}\bar{\Phi}_{c}^{*}(n,y,s) = (-1)^{n+1} \frac{d\bar{\phi}(a,y,s)}{dx} - \frac{d\bar{\phi}(0,y,s)}{dx},$$
(12)

Where

$$\beta_{n} = \frac{n\pi}{a}, \quad \lambda_{n} = \sqrt{(\beta_{n})^{2} + (s/c_{s})^{2}}, \quad (13)$$

It may be shown that the solution to the problem is obtained as follows:

$$\overline{W}_{c}(0,y,s) = e^{-\gamma y} \Big[A_{k,0} \cosh(\mu_{0} y) + B_{k,0} \sinh(\mu_{0} y) \Big],$$

$$\overline{W}_{c}(n,y,s) = e^{-\gamma y} \Big[A_{k,n} \cosh(\mu_{n} y) + B_{k,n} \sinh(\mu_{n} y) \Big],$$

$$\overline{\Phi}_{c}^{*}(0,y,s) = C_{k,0} + D_{k,0} e^{-2\gamma y}, \quad k = 1,2$$
(14)

$$\overline{\Phi}_{c}^{*}(n,y,s) = e^{-2\gamma y} [C_{k,n} \cosh(\delta_{n} y) + D_{k,n} \sinh(\delta_{n} y)], \quad k = 1,2 \quad n = 1,2,...$$

The index k = 1,2 refers to the regions $\eta < y < h$ and $0 < y < \eta$, respectively. In the above equations, the new variable is defined as follows:

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$$\mu_{n} = \sqrt{\lambda_{n}^{2} + \gamma^{2}}, \quad \delta_{n} = \sqrt{\beta_{n}^{2} + \gamma^{2}}, \quad (15)$$

The expressions for $A_{k,n}$, $B_{k,n}$, $C_{k,n}$ and $D_{k,n}$, k = 1, 2, n = 0, 1, 2, ... are unknown functions, which will be obtained from the boundary conditions. With the aid of constitutive equations (6) and (14), it is not difficult to obtain the expressions for the components of stress and electrical displacement in the Laplace transform domain.

$$\overline{\sigma}_{zy}(n, y, s) = e^{\gamma y} \{ kA_{1,n}[\mu_n \sinh(\mu_n y) - \gamma \cosh(\mu_n y)] \\ + k B_{1,n}[\mu_n \cosh(\mu_n y) - \gamma \sinh(\mu_n y)] \\ + e_{150}C_{1,n}[\delta_n \sinh(\delta_n y) - \gamma \cosh(\delta_n y)] \\ + e_{150}D_{1,n}[\delta_n \cosh(\delta_n y) - \gamma \sinh(\delta_n y)] \}$$
(16)

$$\overline{D}_{y}(n,y,s) = -e^{\gamma y} \varepsilon_{110} \{C_{1,n}[\delta_{n} \sinh(\delta_{n} y) - \gamma \cosh(\delta_{n} y)] + D_{1,n}[\delta_{n} \cosh(\delta_{n} y) - \gamma \sinh(\delta_{n} y)]\}, \qquad \zeta < y < h$$

and

$$\overline{\sigma}_{zy}(n, y, s) = e^{\gamma y} \{ k A_{2,n}[\mu_n \sinh(\mu_n y) - \gamma \cosh(\mu_n y)] \\ + k B_{2,n}[\mu_n \cosh(\mu_n y) - \gamma \sinh(\mu_n y)] \\ + e_{150} C_{2,n}[\delta_n \sinh(\delta_n y) - \gamma \cosh(\delta_n y)] \\ + e_{150} D_{2,n}[\delta_n \cosh(\delta_n y) - \gamma \sinh(\delta_n y)] \}$$
(17)

$$\overline{D}_{y}(n, y, s) = -e^{\gamma y} \varepsilon_{110} \{ C_{2,n}[\delta_{n} \sinh(\delta_{n} y) - \gamma \cosh(\delta_{n} y)] + D_{2,n}[\delta_{n} \cosh(\delta_{n} y) - \gamma \sinh(\delta_{n} y)] \} \quad 0 < y < \zeta$$

In this paper, two types of boundary conditions are considered. There are other possible boundary conditions for rectangular planes that are not analyzed in this work. Furthermore, the crack faces are assumed to be not in contact such that the cracks are electrically nonconductive. Therefore, the impermeable condition was adopted. To investigate the transient behavior of cracked FGPRP, we first obtain the dislocation solution of the problem. Using this solution as a Green's function, the desired field quantities may easily be obtained.

2.1 The plane is free on two opposite edges and fixed on the other edges (Problem I)

Consider an electro-elastic dislocation located at an arbitrary position (η, ξ) in a functionally graded rectangular piezoelectric plate with length *a* and width *h*. (Fig. 1).



Figure 1 Schematic view of the FGPRP with a screw dislocation (C-F-C-F).

Referring to Fig. 1, the boundary and traction-free conditions of the problem may be expressed as follows:

$$\sigma_{zx}(0,y,t) = \sigma_{zx}(a,y,t) = 0 \qquad 0 < y < h$$

$$D_{x}(0,y,t) = D_{x}(a,y,t) = 0 \qquad 0 < y < h$$

$$w(x,0,t) = w(x,h,t) = 0 \qquad 0 < x < a$$
(18)

$$\varphi(x,0,t) = \varphi(x,h,t) = 0$$
 $0 < x < a$

The continuity conditions for the FGPRP together with equations representing electro-elastic dislocation are represented by:

$$\sigma_{zy}(x,\zeta^{-},t) = \sigma_{zy}(x,\zeta^{+},t)$$

$$D_{y}(x,\zeta^{-},t) = D_{y}(x,\zeta^{+},t)$$

$$w(x,\xi^{+},t) - w(x,\xi^{-},t) = b_{wz}(t)H(x-\eta)$$

$$\phi(x,\xi^{+},t) - \phi(x,\xi^{-},t) = b_{\varphi z}(t)H(x-\eta)$$
(19)

where H(.) is the Heaviside step function and $b_{wz}, b_{\varphi z}$ are the Burgers vectors. Based on the Eqs. (5) and (19), it can be seen that:

$$w(x,\zeta^{+},t) - w(x,\zeta^{-},t) = b_{wz}(t)H(x-\eta)$$

$$\overline{\varphi}(x,\zeta^{+},t) - \overline{\varphi}(x,\zeta^{-},t) = \left[b_{\varphi z}(t) - \alpha b_{wz}(t)\right]H(x-\eta)$$
(20)

The finite Fourier cosine transform of Eqs. (20), in view of continuity conditions in Eq. (19) reduce to:

$$\overline{W}_{c}(n,\zeta^{+},s) - \overline{W}_{c}(n,\zeta^{-},s) = -\frac{b_{wz}(s)}{\beta_{n}}sin(\beta_{n}\eta)$$

$$\overline{\Phi}_{c}(n,\zeta^{+},s) - \overline{\Phi}_{c}(n,\zeta^{-},s) = -\frac{b_{\varphi z}(s) - \alpha b_{wz}(s)}{\beta_{n}}sin(\beta_{n}\eta)$$

$$\frac{d\overline{W}_{c}(n,\zeta^{+},s)}{dy} = \frac{d\overline{W}_{c}(n,\zeta^{-},s)}{dy}$$

$$\frac{d\overline{\Phi}_{c}(n,\zeta^{+},s)}{dy} = \frac{d\overline{\Phi}_{c}(n,\zeta^{-},s)}{dy}$$
(21)

After some relatively straight manipulations, we could find the unknown functions in Eq. (14) which are defined in Appendix A. The corresponding stress and electric displacement components may be obtained with the help of Eqs. (16) and (17):

$$\begin{split} \bar{\sigma}_{zy}(x,y,s) &= -\frac{2\gamma e_{150}\eta\alpha}{a(1-e^{-2\gamma h})} b_{wz}(s) + k\eta e^{\gamma(\zeta+y)} [\mu_0 \cosh[\mu_0(h-\zeta)] + \gamma \sinh[\mu_0(h-\zeta)]] \\ &\times \frac{[\mu_0 \cosh(\mu_0 y) - \gamma \sinh(\mu_0 y)]}{a\mu_0 \sinh(\mu_0 \zeta) \sinh(\mu_0 h)} b_{wz}(s) + \frac{2\gamma e_{150}\eta}{a(1-e^{-2\gamma h})} b_{\varphi z}(s) \\ &+ \frac{2kb_{wz}(s)e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\mu_n, h-\zeta) E_{2n}(\mu_n, y) \sin(\beta_n \eta) \cos(\beta_n x) \\ &- \frac{2e_{150}\alpha b_{wz}(s)e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_n, h-\zeta) E_{2n}(\delta_n, y) \sin(\beta_n \eta) \cos(\beta_n x) \\ &+ \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_n, h-\zeta) E_{2n}(\delta_n, y) \sin(\beta_n \eta) \cos(\beta_n x) \\ &+ \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_n, h-\zeta) E_{2n}(\delta_n, y) \sin(\beta_n \eta) \cos(\beta_n x) \\ &+ \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_n, h-\zeta) E_{2n}(\delta_n, y) \sin(\beta_n \eta) \cos(\beta_n x) \\ &+ \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_n, h-\zeta) E_{2n}(\delta_n, y) \sin(\beta_n \eta) \cos(\beta_n x) \\ &+ \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_n, h-\zeta) E_{2n}(\delta_n, y) \sin(\beta_n \eta) \cos(\beta_n x) \\ &+ \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_n, h-\zeta) E_{2n}(\delta_n, y) \sin(\beta_n \eta) \cos(\beta_n x) \\ &+ \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_n, h-\zeta) E_{2n}(\delta_n, y) \sin(\beta_n \eta) \cos(\beta_n x) \\ &+ \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_n, h-\zeta) E_{2n}(\delta_n, y) \sin(\beta_n \eta) \cos(\beta_n x) \\ &+ \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_n, h-\zeta) E_{2n}(\delta_n, y) \sin(\beta_n \eta) \cos(\beta_n x) \\ &+ \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_n, h-\zeta) E_{2n}(\delta_n, y) \sin(\beta_n \eta) \cos(\beta_n x) \\ &+ \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_n, h-\zeta) E_{2n}(\delta_n, y) \sin(\beta_n \eta) \cos(\beta_n x) \\ &+ \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_n, h-\zeta) E_{2n}(\delta_n, y) \sin(\beta_n \eta) \cos(\beta_n x) \\ &+ \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_n, h-\zeta) E_{2n}(\delta_n, y) \sin(\beta_n \eta) \cos(\beta_n x) \\ &+ \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_n, h-\zeta) E_{2n}(\delta_n, y) \sin(\beta_n \eta) \cos(\beta_n x) \\ &+ \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_n, h-\zeta) E_{2n}(\delta_n, y) \sin(\beta_n \eta) \cos(\beta_n x) \\ &+ \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_n, h-\zeta) E_{2n}(\delta_n, y) \sin(\beta$$

$$\begin{split} \bar{\sigma}_{zy}(x,y,s) &= -\frac{2\eta\gamma e_{150}\alpha b_{wz}(s)}{a(1-e^{-2\gamma h})} + k\eta e^{\gamma(\zeta+y)} [\mu_{0}\cosh(\mu_{0}\zeta) - \gamma\sinh(\mu_{0}\zeta)] \\ &\times \frac{\{\mu_{0}\cosh[\mu_{0}(h-y)] + \gamma\sinh[\mu_{0}(h-y)]\}b_{wz}(s)}{a\mu_{0}\sinh(\mu_{0}\zeta)\sinh(\mu_{0}h)} + \frac{2\eta\gamma e_{150}b_{\varphi z}(s)}{a(1-e^{-2\gamma h})} \end{split}$$
(22)
$$-\frac{2kb_{wz}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} F_{1n}(\mu_{n},\zeta)E_{1n}(\mu_{n},h-y)\sin(\beta_{n}\eta)\cos(\beta_{n}x) \\ &+ \frac{2\alpha b_{wz}(s)e_{150}e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} F_{1n}(\delta_{n},\zeta)E_{1n}(\delta_{n},h-y)\sin(\beta_{n}\eta)\cos(\beta_{n}x) \\ &- \frac{2b_{\varphi z}(s)e_{150}e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} F_{1n}(\delta_{n},\xi)E_{1n}(\delta_{n},h-y)\sin(\beta_{n}\eta)\cos(\beta_{n}x) \\ &- \frac{2b_{\varphi z}(s)e_{10}e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} F_{1n}(\delta_{n},\xi)E_{1n}(\delta_{n},h-y)\sin(\beta_{n}\eta)\cos(\beta_{n}x) \\ &- \frac{2b_{\varphi z}(s)e_{10}e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} F_{1n}(\delta_{n},\xi)E_{1n}(\delta_{n},h-y)\sin(\beta_{n}\eta)\cos(\beta_{n}y) \\ &- \frac{2b_{\varphi z}(s)e_{10}e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} F_{1n}(\delta_{n},\xi)E_{1n}(\delta_{n},h-y)\sin(\beta_{n}y)\cos(\beta_{n}y) \\ &- \frac{2b_{\varphi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} F_{1n}(\delta_{n},\xi)E_$$

and

$$\begin{split} \overline{D}_{y}(x,y,s) &= \frac{2\gamma e_{150} \eta b_{wz}(s)}{a(1-e^{-2\gamma h})} - \frac{2\gamma \epsilon_{110} \eta b_{\varphi z}(s)}{a(1-e^{-2\gamma h})} \\ &+ \frac{2e_{150} b_{wz}(s) e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_{n},h-\zeta) E_{2n}(\delta_{n},y) \sin(\beta_{n}\eta) \cos(\beta_{n}x) \\ &- \frac{2\epsilon_{110} b_{\varphi z}(s) e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_{n},h-\zeta) E_{2n}(\delta_{n},y) \sin(\beta_{n}\eta) \cos(\beta_{n}x) \qquad 0 < y < \zeta \end{split}$$

$$\begin{split} \overline{D}_{y}(x,y,s) &= \frac{2\gamma e_{150} \eta b_{wz}(s)}{a(1-e^{-2\gamma h})} - \frac{2\gamma \epsilon_{110} \eta b_{\varphi z}(s)}{a(1-e^{-2\gamma h})} \\ &+ \frac{2e_{150} b_{wz}(s) e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_{n},h-y) E_{2n}(\delta_{n},\xi) \sin(\beta_{n}\eta) \cos(\beta_{n}x) \\ &- \frac{2\epsilon_{110} b_{\varphi z}(s) e^{\gamma(\zeta+y)}}{\pi} \sum_{n=1}^{\infty} F_{2n}(\delta_{n},h-y) E_{2n}(\delta_{n},\xi) \sin(\beta_{n}\eta) \cos(\beta_{n}x) \quad \zeta < y < h \end{split}$$

Where

$$F_{1n}(p_{1},p_{2}) = \frac{\gamma \sinh(p_{1}p_{2}) - p_{1}\cosh(p_{1}p_{2})}{np_{1}\sinh(p_{1}h)}$$

$$E_{1n}(q_{1},q_{2}) = \gamma \sinh(q_{1}q_{2}) + q_{1}\cosh(q_{1}q_{2})$$

$$F_{2n}(p_{1},p_{2}) = \frac{\gamma \sinh(p_{1}p_{2}) + p_{1}\cosh(p_{1}p_{2})}{np_{1}\sinh(p_{1}h)}$$

$$E_{n}(q_{1},q_{2}) = q_{1}\cosh(q_{1}q_{2}) - \gamma \sinh(q_{1}q_{2})]$$
(24)

From (22) and (23), it is seen that large values of n the series solutions converge slowly and a large number of terms are required to obtain accurate results. To circumvent this difficulty, the series in (22) and (23) should be performed differently. To this end, we consider the following relation

$$\begin{split} \sum_{n=1}^{\Psi} F_{1n}(\mu_{n},\zeta) E_{1n}(\mu_{n},h-y) \sin(\beta_{n}\eta) \cos(\beta_{n}x) = \\ \sum_{n=1}^{\Psi} \frac{\mu_{n}(\sin[\beta_{n}(x+\eta)] - \sin[\beta_{n}(x-\eta)])}{4n(1 - e^{-2\mu_{n}h})} e^{-\mu_{n}(y-\xi)} \\ + \gamma^{2} \sum_{n=1}^{\Psi} \frac{\sin(\beta_{n}\eta) \cos(\beta_{n}x)}{2n\mu_{n}(1 - e^{-2\mu_{n}h})} e^{\mu_{n}(\zeta-y)} \\ + \sum_{n=1}^{\Psi} \frac{(\gamma^{2} + 2\gamma\mu_{n} - (\mu_{n})^{2}) \sin(\beta_{n}\eta) \cos(\beta_{n}x)}{2n\mu_{n}(1 - e^{-2\mu_{n}h})} e^{-\mu_{n}(\zeta+y)} \\ - \sum_{n=1}^{\Psi} \frac{(\gamma - \mu_{n})^{2} \sin(\beta_{n}\eta) \cos(\beta_{n}x)}{2n\mu_{n}(1 - e^{-2\mu_{n}h})} e^{\mu_{n}(-2h+\zeta+y)} \\ + \sum_{n=1}^{\Psi} \frac{(\gamma^{2} + 2\gamma\mu_{n} - (\mu_{n})^{2}) \sin(\beta_{n}\eta) \cos(\beta_{n}x)}{2n\mu_{n}(1 - e^{-2\mu_{n}h})} e^{\mu_{n}(-2h+\zeta+y)} \\ \end{split}$$

For a large value of n, By using the following relation

$$\sum_{n=1}^{\infty} e^{-\beta_n(y-\xi)} \sin[\beta_n(x-\eta)] = \frac{0.5 \sin[\frac{\pi}{a}(x-\eta)]}{\cosh[\frac{\pi}{a}(y-\xi)] - \cos[\frac{\pi}{a}(x-\eta)]}$$
(26)

After some algebra, Eq. (25) can be expressed in a more suitable form as

$$\begin{split} \sum_{n=1}^{\Psi} F_{1n}(\mu_{n},\zeta) E_{1n}(\mu_{n},h-y) \sin(\beta_{n}\eta) \cos(\beta_{n}x) &= (\frac{\pi}{8a}) \times \\ \{ \frac{\sin[\frac{\pi(x-\eta)}{a}]}{(\cosh[\frac{\pi(y-\xi)}{a}] - \cos[\frac{\pi(x-\eta)}{a}]} \\ - \frac{\sin[\frac{\pi(x+\eta)}{a}]}{(\cosh[\frac{\pi(y-\xi)}{a}] - \cos[\frac{\pi(x+\eta)}{a}]} \} \\ + \frac{1}{4} \sum_{n=1}^{\Psi} (\frac{\mu_{n}e^{\mu_{n}(y-\xi)}}{n(1-e^{-2\mu_{n}h})} - \frac{\pi}{a} e^{-\beta_{n}(y-\xi)}) (\sin[\beta_{n}(x+\eta)] - \sin[\beta_{n}(x-\eta)]) \\ + \gamma^{2} \sum_{n=1}^{\Psi} \frac{\sin(\beta_{n}\eta)\cos(\beta_{n}x)}{2n\mu_{n}(1-e^{-2\mu_{n}h})} e^{\mu_{n}(\xi-y)} \\ + \sum_{n=1}^{\Psi} \frac{(\gamma^{2} + 2\gamma\mu_{n} - (\mu_{n})^{2})\sin(\beta_{n}\eta)\cos(\beta_{n}x)}{2n\mu_{n}(1-e^{-2\mu_{n}h})} e^{\mu_{n}(-2h+y-\xi)} \\ - \sum_{n=1}^{\Psi} \frac{(\gamma^{2} + 2\gamma\mu_{n} - (\mu_{n})^{2})\sin(\beta_{n}\eta)\cos(\beta_{n}x)}{2n\mu_{n}(1-e^{-2\mu_{n}h})} e^{\mu_{n}(-2h+y-\xi)} \\ + \sum_{n=1}^{\Psi} \frac{(\gamma^{2} + 2\gamma\mu_{n} - (\mu_{n})^{2})\sin(\beta_{n}\eta)\cos(\beta_{n}x)}{2n(1-e^{-2\mu_{n}h})} e^{\mu_{n}(-2h+y-\xi)} \\ = \xi < y < h \end{split}$$

It is obvious that the relations appearing in (27) converge sufficiently rapidly for a large value n, which makes the summations susceptible to numerical evaluation. A procedure analogous to relation (25) is used to obtain suitable forms for other similar terms in the field components. In all the problems discussed here, it is not difficult to show the singularity of field quantities. Also, It can be shown that the first term of Eq. (27) has a Cauchy-type singularity as $x \rightarrow \eta$, $y \rightarrow \xi$. Faal and Dehgan [11].

2.2 The rectangular plane is fixed on one edge and free on all other edges (Problem II)

To solve the present problem, the jumps of the displacement and the electric potential across the dislocation line are held, and the edge conditions can be defined as follows:

$$\sigma_{zx}(0,y,t) = 0, \quad \sigma_{zx}(a,y,t) = 0, \quad 0 < y < h$$

$$D_{x}(0,y,t) = 0, \quad D_{x}(a,y,t) = 0, \quad 0 < y < h$$

$$\sigma_{zy}(x,h,t) = 0, \quad D_{y}(x,h,t) = 0, \quad 0 < x < a$$
(28)

$$w(x,0,t) = 0, \phi(x,0,t) = 0, \quad 0 < x < a$$

Applying the finite Fourier cosine and Laplace transforms to Eqs. (28), in view of continuity conditions, the stress and electric field components for this problem are obtained as follows:

$$\begin{split} \overline{\sigma}_{zy}(x,y,s) &= -\frac{k\eta e^{\gamma(y+\zeta)}(s/c_s)\sinh[\mu_0(h-\zeta)][\mu_0\cosh(\mu_0 y) - \gamma\sinh(\mu_0 y)]b_{wz}(s)}{a[\gamma\mu_0\sinh(\mu_0 h) - {\mu_0}^2\cosh(\mu_0 h)]} \\ &- \frac{2kb_{wz}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \lambda_n^2 F_{3n}(\mu_n, h-\xi) E_{3n}(\mu_n, y)\sin(\beta_n \eta)\cos(\beta_n x) \\ &+ \frac{2e_{150}\alpha b_{wz}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_n^2 F_{3n}(\delta_n, h-\xi) E_{3n}(\delta_n, y)\sin(\beta_n \eta)\cos(\beta_n x) \\ &- \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_n^2 F_{3n}(\delta_n, h-\xi) E_{3n}(\delta_n, y)\sin(\beta_n \eta)\cos(\beta_n x) \\ &- \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_n^2 F_{3n}(\delta_n, h-\xi) E_{3n}(\delta_n, y)\sin(\beta_n \eta)\cos(\beta_n x) \\ &- \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_n^2 F_{3n}(\delta_n, h-\xi) E_{3n}(\delta_n, y)\sin(\beta_n \eta)\cos(\beta_n x) \\ &- \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_n^2 F_{3n}(\delta_n, h-\xi) E_{3n}(\delta_n, y)\sin(\beta_n \eta)\cos(\beta_n x) \\ &- \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_n^2 F_{3n}(\delta_n, h-\xi) E_{3n}(\delta_n, y)\sin(\beta_n \eta)\cos(\beta_n x) \\ &- \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_n^2 F_{3n}(\delta_n, h-\xi) E_{3n}(\delta_n, y)\sin(\beta_n \eta)\cos(\beta_n x) \\ &- \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_n^2 F_{3n}(\delta_n, h-\xi) E_{3n}(\delta_n, y)\sin(\beta_n \eta)\cos(\beta_n x) \\ &- \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_n^2 F_{3n}(\delta_n, h-\xi) E_{3n}(\delta_n, y)\sin(\beta_n \eta)\cos(\beta_n x) \\ &- \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_n^2 F_{3n}(\delta_n, h-\xi) E_{3n}(\delta_n, y)\sin(\beta_n \eta)\cos(\beta_n x) \\ &- \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_n^2 F_{3n}(\delta_n, h-\xi) E_{3n}(\delta_n, y)\sin(\beta_n \eta)\cos(\beta_n x) \\ &- \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_n^2 F_{3n}(\delta_n, h-\xi) E_{3n}(\delta_n, y)\sin(\beta_n \eta)\cos(\beta_n x) \\ &- \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_n^2 F_{3n}(\delta_n, h-\xi) E_{3n}(\delta_n, y)\sin(\beta_n \eta)\cos(\beta_n x) \\ &- \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_n^2 F_{3n}(\delta_n, h-\xi) E_{3n}(\delta_n, y)\sin(\beta_n \eta)\cos(\beta_n x) \\ &- \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_n^2 F_{3n}(\delta_n, h-\xi) E_{3n}(\delta_n, y)\sin(\beta_n \eta)\cos(\beta_n x) \\ &- \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_n^2 F_{3n}(\delta_n, h-\xi) E_{3n}(\delta_n, y)\sin(\beta_n \eta)\cos(\beta_n x) \\ &- \frac{2e_{150}b_{\varphi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_n^2 F_{3n}(\delta_n, h-\xi) E_{3n}(\delta_n, y)\sin(\beta_n y)\cos(\beta_n x) \\ &- \frac{2e_{150}$$



Figure 2 Functionally graded piezoelectric rectangular plate with a screw dislocation (C-F-F-F).

$$\begin{split} \overline{\sigma}_{zy}(x,y,s) &= -\frac{k\eta s e^{\gamma(y+\zeta)} sinh[\mu_{0}(h-y)][\mu_{0} cosh(\mu_{0}\zeta) - \gamma sinh(\mu_{0}\zeta)]b_{wz}(s)}{ac_{s}\mu_{0}[\gamma sinh(\mu_{0}h) - \mu_{0} cosh(\mu_{0}h)]} \\ &- \frac{2kb_{wz}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \lambda_{n}^{2} F_{3n}(\mu_{n},h-y) E_{3n}(\mu_{n},\zeta) sin(\beta_{n}\eta) cos(\beta_{n}x) \\ &+ \frac{2e_{150}\alpha b_{wz}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_{n}^{2} F_{3n}(\delta_{n},h-y) E_{3n}(\delta_{n},\zeta) sin(\beta_{n}\eta) cos(\beta_{n}x) \\ &- \frac{2e_{150}b_{\phi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_{n}^{2} F_{3n}(\delta_{n},h-y) E_{3n}(\delta_{n},\zeta) sin(\beta_{n}\eta) cos(\beta_{n}x) \\ &- \frac{2e_{150}b_{\phi z}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_{n}^{2} F_{3n}(\delta_{n},h-y) E_{3n}(\delta_{n},\zeta) sin(\beta_{n}\eta) cos(\beta_{n}x) \\ &\zeta < y < h \end{split}$$

and

$$\overline{D}_{y}(x,y,s) = -\frac{2e_{150}b_{wz}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_{n}^{2}F_{3n}(\delta_{n},h-\xi)E_{3n}(\delta_{n},y)\sin(\beta_{n}\eta)\cos(\beta_{n}x) + \frac{2b_{\varphi z}(s)\epsilon_{110}e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_{n}^{2}F_{3n}(\delta_{n},h-\xi)E_{3n}(\delta_{n},y)\sin(\beta_{n}\eta)\cos(\beta_{n}x) \qquad 0 < y < \zeta$$
(30)

$$\overline{D}_{y}(x,y,s) = -\frac{2e_{150}b_{wz}(s)e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_{n}^{2}F_{3n}(\delta_{n},h-y)E_{3n}(\delta_{n},\xi)\sin(\beta_{n}\eta)\cos(\beta_{n}x)$$
$$+\frac{2b_{\varphi z}(s)\varepsilon_{110}e^{\gamma(y+\zeta)}}{\pi} \sum_{n=1}^{\infty} \beta_{n}^{2}F_{3n}(\delta_{n},h-y)E_{3n}(\delta_{n},\xi)\sin(\beta_{n}\eta)\cos(\beta_{n}x) \qquad \zeta < y < h$$

and the functions F_{3n} and E_{3n} are given by

$$F_{3n}(p_1,p_2) = \sinh(p_1p_2)$$

$$E_{3n}(q_1,q_2) = \frac{q_1 \cosh(\mu_n q_2) - \gamma \sinh(\mu_n q_2)}{nq_1(\gamma \sinh(q_1 h) - q_1 \cosh(q_1 h))}$$
(31)

Similar to the previous problems, it is not difficult to obtain suitable expressions for the components of the stress and electric displacement in the Laplace transform domain for a large value \mathbf{n} . This task is taken up as follows:

$$\sum_{n=1}^{\infty} \lambda_{n}^{2} F_{3n}(\mu_{n}, h-\xi) E_{3n}(\mu_{n}, y) \sin(\beta_{n}\eta) \cos(\beta_{n}x) =$$

$$\sum_{n=1}^{\infty} \frac{\lambda_{n}^{2} (1-e^{-2\mu_{n}(h-y)}) [\mu_{n}(1+e^{-2\mu_{n}\zeta})-\gamma(1-e^{-2\mu_{n}\zeta})]}{4n\mu_{n}[\gamma(1-e^{-2\mu_{n}h})-\mu_{n}(1+e^{-2\mu_{n}h})]}$$

$$\times e^{\mu_{n}(\xi-y)} (\sin[\beta_{n}(x+\eta)] - \sin[\beta_{n}(x-\eta)])$$
(32)

after a somewhat lengthy but routine analysis, the following result was obtained:

$$\begin{split} &\sum_{n=1}^{\infty} \lambda_{n}^{2} F_{3n}(\mu_{n},h-\xi) E_{3n}(\mu_{n},y) \sin(\beta_{n}\eta) \cos(\beta_{n}x) = -\frac{\pi}{8a} \times \\ &\left[\frac{\sin[\frac{\pi}{a}(x+\eta)]}{\cosh[\frac{\pi}{a}(y-\xi)] - \cos[\frac{\pi}{a}(x+\eta)]} - \frac{\sin[\frac{\pi}{a}(x-\eta)]}{\cosh[\frac{\pi}{a}(y-\xi)] - \cos[\frac{\pi}{a}(x-\eta)]} \right] \\ &+ \frac{1}{4} \sum_{n=1}^{\infty} \left[\frac{\lambda_{n}^{2}(1-e^{-2\mu_{n}(h-y)})(\mu_{n}(1+e^{-2\mu_{n}\xi}) - \gamma(1-e^{-2\mu_{n}\xi}))e^{-\mu_{n}(y-\xi)}}{n\mu_{n}[\gamma(1-e^{-2\mu_{n}h}) - \mu_{n}(1+e^{-2\mu_{n}h})]} + \frac{\pi}{a} e^{-\frac{n\pi}{a}(y-\xi)} \right] \\ &\times (\sin[\beta_{n}(x+\eta)] - \sin[\beta_{n}(x-\eta)]) \end{split}$$
(33)

Proceeding the same way for other similar terms in Eqs. (29) and (30), one can derive similar suitable forms for the summation terms which are appeared in the field components. Since the resulting infinite series could be summed in closed form. Hence, it is not difficult to study the singular behavior of the kernels. functionally graded rectangular plane.

It is possible to verify the stress components for the case of functionally graded rectangular plane with no piezoelectric effect. In the particular case of an isotropic elastic rectangular plane, by applying $e_{150} = 0$, and $\varepsilon_{110} = 0$, the stress components (22) reduce to

$$\sigma_{zy}(x,y) = \frac{2b_{wz}\eta c_{440}\gamma e^{2\gamma h}}{a(e^{2\gamma h} - 1)} - \frac{2b_{wz}c_{440}e^{[\gamma(y+\xi)]}}{\pi} \sum_{n=1}^{\infty} G_n H_{1n}[\gamma \sinh(\theta_n(h-y)) + \theta_n \cosh(\theta_n(h-y))]\cos(\frac{n\pi x}{a})$$

$$\sigma_{zx}(x,y) = -\frac{2b_{wz}c_{440}e^{[\gamma(y+\xi)]}}{a} \sum_{n=1}^{\infty} nG_n H_{1n}\sinh(\theta_n(h-y))\sin(\frac{n\pi x}{a}) \qquad 0 < y < \zeta$$

$$\sigma_{zy}(x,y) = \frac{2b_{wz}\eta c_{440}\gamma e^{2\gamma h}}{a(e^{2\gamma h}-1)} + \frac{2b_{wz}c_{440}e^{[\gamma(y+\xi)]}}{\pi}\sum_{n=1}^{\infty}G_nH_{2n}[-\gamma\sinh(\theta_n y) + \theta_n\cosh(\theta_n y)]\cos(\frac{n\pi x}{a})$$

$$\sigma_{zx}(x,y) = -\frac{2b_{wz}c_{440}e^{[\gamma(y+\xi)]}}{a} \sum_{n=1}^{\infty} nG_n H_{2n} \sinh(\theta_n y) \sin(\frac{n\pi x}{a}) \qquad \zeta < y < h$$
(34)

where

$$G_{n} = \frac{\sin(\frac{n\pi\eta}{a})}{n\theta_{n}\sinh(\theta_{n}h)},$$

$$H_{1n} = \gamma \sinh(\theta_{n}\xi) - \theta_{n}\cosh(\theta_{n}\xi),$$

$$H_{2n} = \gamma \sinh[\theta_{n}(h-\xi)] + \theta_{n}\cosh[\theta_{n}(h-\xi)]$$

and

$$\theta_{n} = \sqrt{\gamma^{2} + (\frac{n\pi}{a})^{2}}$$
(35)

The results are coincident with those reported in the article of Faal and Dehgan [11].

3 Multiple cracks formulation

The integral equations of the problems can then be derived by using the dislocation solution as Green's function. Thus, the dislocation solution may be utilized for analyzing the aforementioned cracked structures subject to impact loads. The geometry of a crack may be described in parametric form, as

where (x_{0i}, y_{0i}) is the location of the center and L_i is the half-length of the crack. The components of traction and electric displacements on the boundary of the *i*th crack caused by the continuous distribution of dislocations with density $B_{ki}(q, s)$, $k \in \{w, \varphi\}$ are given, from (22), (23),(29), and (30), by

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$$\overline{\sigma}_{zn}(x_{i}(p), y_{i}(p), s) = \sum_{j=1}^{N} \int_{-1}^{1} [K_{ij}^{11}(p, q, s)B_{vzj}(q, s) + K_{ij}^{12}(p, q, s)B_{\phi j}(q, s)]L_{j}dq$$

$$\overline{D}_{n}(x_{i}(p), y_{i}(p), s) = \sum_{j=1}^{N} \int_{-1}^{1} [K_{ij}^{21}(p, q, s)B_{vzj}(q) + K_{ij}^{22}(p, q, s)B_{\phi j}(q)]L_{j}dq$$
(37)

where
$$L_j(q) = \sqrt{[x'_j(q)]^2 + [y'_j(q)]^2}$$
. The left-hand side of equations (37), with opposite signs, is
the traction and electric displacement induced by the external load on the presumed crack
surfaces in the intact FGPRP. For the internal cracks, the displacement and electric potential
discontinuity across the *j*th crack are:

$$\overline{w}_{j}^{-}(\mathbf{p},\mathbf{s}) - \overline{w}_{j}^{+}(\mathbf{p},\mathbf{s}) = \int_{-1}^{p} B_{wzj}(\mathbf{q},\mathbf{s}) L_{j} dq$$

$$\overline{\phi}_{j}^{-}(\mathbf{p},\mathbf{s}) - \overline{\phi}_{j}^{+}(\mathbf{p},\mathbf{s}) = \int_{-1}^{p} B_{\phi j}(\mathbf{q},\mathbf{s}) L_{j} dq$$
(38)

The sides' conditions are required to render the problem determinate. In this case, the single-valuedness condition becomes

$$\int_{-1}^{1} B_{kj}(q,s) L_{j} dq = 0 \qquad k \in \{wz, \phi\}$$
(39)

It may be shown that the kernel of integral equations (37) has only Cauchy-type singularity. Hence, the dislocation densities on the surface of cracks are taken as

$$B_{kj}(q,s) = \frac{g_{kj}(q,s)}{\sqrt{1-q^2}}, \ -1 \le q \le 1, \ k \in \{wz, \phi\}$$
(40)

The unknown functions $g_{kj}(q,s)$ are bounded. Substitution of Eq. (40) into Eqs. (39) and (37) and carrying out the numerical technique for the solution of Cauchy singular integral equations leads to the Laplace transform of the dislocation density function devised by Erdogan et al. [15]. The stress intensity factors at the tips of an embedded crack become:

$$K_{Li}(s) = \frac{\sqrt{L_{j}(-1)}}{2} [c_{44}(y_{Li})g_{wzi}(-1,s) + e_{15}(y_{Li})g_{\phi i}(-1,s)]$$

$$K_{Ri}(s) = -\frac{\sqrt{L_{j}(1)}}{2} [c_{44}(y_{Ri})g_{wzi}(1,s) + e_{15}(y_{Ri})g_{\phi i}(1,s)]$$
(41)

For brevity, the details of the derivation of stress intensity factors are not given here. The inverse Laplace transform of DSIFs is accomplished numerically by Stehfest's method Cohen [16].

4 Numerical results and discussion

The validity of formulation is examined by considering the problem of a functionally graded rectangular plane containing a vertical central crack under the static point load, Figure 3a, which is solved by Faal and Dehghan [11] is re-examined and the results are shown in Figures (3b and 3b). As it may be observed, the agreement of the results in the above examples is reasonable. Furthermore, the results for the static case are compared with those presented for the problem by Faal and Dehghan [11].



Figure 3a A functionally graded rectangular plane with a vertical crack under point load.



Figure 3b Variations of stress intensity factor with crack location for a functionally graded rectangular plane weakened by a vertical crack under point load.



Figure 3c Variations of stress intensity factor with x_c/a for a functionally graded rectangular plane weakened by a vertical crack under point load.

d/h	-	2L/h = 0.1	2L/h = 0.2	2L/h = 0.3	2L/h = 0.4
0.95	Present Study	1.099	1.285	1.542	1.914
	Ref. [11]	1.097	1.284	1.531	1.913
0.9	Present Study	1.035	1.130	1.272	1.489
	Ref. [11]	1.033	1.128	1.272	1.487
0.75	Present Study	1.010	1.059	1.136	1.268
	Ref. [11]	1.011	1.059	1.136	1.269
0.5	Present Study	1.007	1.049	1.123	1.243
	Ref. [11]	1.009	1.048	1.120	1.242

Table1 Stress intensity factors for a = h.

The values of SIFs, are given in Table (1) for various crack locations and lengths in an elastic rectangular finite plane. The results are in excellent agreement with those obtained by Faal and Dehghan. Therefore, the validity of the present crack formulation is confirmed.

This section deals with the results of several numerical examples that may have some physical importance and will be discussed. The FGPRP is subjected to uniform electromechanical impact loadings $\sigma_{zy} = \tau_0 H(t)$ and $D_y = D_0 H(t)$ which are applied at crack faces. To reflect the combination between the mechanical impact and electric impact, the electromechanical coupling factor is introduced $\lambda_D = D_0 e_{150} / \tau_0 \varepsilon_{110}$.

In this paper, impermeable conditions prevail and unless otherwise stated, the geometry of the rectangular plane and electromechanical coupling factor in the ensuing examples, are identified as a = 0.02(m), h = 0.01(m) and $\lambda_D = 1.0$, respectively. Furthermore, DSIFs are normalized by $K_0 = \tau_0 / \sqrt{L}$, where *L* is the half-length of the crack. The numerical example to be considered here is a PZT-4 rectangular plane. Wang et al. [17]. The material constants for PZT-4 are

$$c_{440} = 2.56 \times 10^{10} \frac{N}{m^2}, e_{150} = 12.7 \frac{C}{m^2}, \epsilon_{110} = 6.4634 \times 10^{-9} \frac{C}{Vm^2}, \rho = 7500 \frac{kg}{m^3}$$
(42)

In the first example, a central crack with three different FG exponents $\gamma h = 0.5$, $\gamma h = 1.0$ and $\gamma h = 2.0$ is considered for both types of boundary conditions (Figures 4a, 4b).



Figure 4 A central crack in functionally graded piezoelectric rectangular plane (FGPRP): (a) C-F-C-F case and (b) C-F-F-F case.



Figure 5 Variation of stress dynamic stress intensity factor of a crack in a FGPRP with different FG parameter: (a) C-F-C-F case and C-F-F-F case.

The variations of normalized DSIFs for a crack are depicted in Figures 5a. and 5b. The results in Figures 5a. and 5b indicate that the value of the DSIF increases quickly with time, reaches a peak value, and then decreases oscillating around the corresponding static value. The peak value of the DSIF will increase as the FG exponent γh increases for both types of boundary conditions. Moreover, the FG exponent does not alter the time of the maxima of DSIFs in the C-F-C-F case. Furthermore, the boundary effect is significant, and for the C-F-C-F case, the peak values of the DSIFs are larger than that of the C-F-F-F case. In the case of C-F-F-F, the time of occurrence of the maxima of DSIFs increases gradually with increase γh .

The variations of DSIFs of a crack with length 2L/h = 0.6 located at three different locations, from the lower boundary, are shown, in Figures 6a and 6b. It is seen that the increase y_c/h results in an increase in the DSIF for the C-F-C-F case whereas, it tends to decrease for the C-F-F case.



Figure 6 Variation of stress dynamic stress intensity factor of a crack in a FGPRP at three different locations: (a) C-F-C-F case and C-F-F-F case.

Nevertheless, the effect of crack location on values of the maxima of DSIFs is not significant for the C-F-F-F case (Figure 6b).

The variations of the DSIFs under the various value a/L are plotted for a central crack under impact electromechanical load (Figure7a and 7b). It can be seen that the boundary effect is significant. For both two cases, the peak value DSIFs decreases as the length of the FGPRP increases, and the crack becomes harder to propagate. A much longer time elapses for DSIF to reach its static value for the C-F-F-F case to the C-F-C-F case. From these figures, we can observe the presence of the dynamic overshoot phenomenon. Moreover, the phenomenon is intensified with the decrease a/L.



Figure 7 Variation of stress dynamic stress intensity factor of a crack in a FGPRP with different plane length: (a) C-F-C-F case and C-F-F-F case.

The variations of the normalized DSIF with the normalized time for two types of boundary conditions at different values of electromechanical coupling factor are plotted in Figures 8a and 8b. All the curves reach a maximum and then oscillate with decreasing peaks. For both types of boundary conditions, the effect of the coupling factor λ_D is more significant. This implies that the large value of the electromechanical coupling factor will enhance the crack initiation, while the small value will retard the crack initiation for the C-F-C-F and C-F-F-F cases.



Figure 8 Variation of stress dynamic stress intensity factor of a crack in a FGPRP with different electromechanical coupling factor: (a) C-F-C-F case and C-F-F-F case.

The formulation may be employed for the analysis of FGPRP containing multiple cracks. To this end, the study of the interaction between two collinear cracks is taken up, Figures 9a and 9b. Referring to Figures 10a and 10b, the DSIF increases rapidly from zero to a peak value well above its corresponding static value and then oscillates about it. The cracks distance d/L was selected to be 1/3, 1/6 and 1/12 to ensure that interaction effects exist between these central cracks. We observe that interaction between cracks enhances the DSIFs of crack which is more pronounced at R_1 and L_2 because it is closer to the other crack.



Figure 9 Two collinear cracks in functionally graded piezoelectric rectangular plane: (a) C-F-C-F case and C-F-F-F case.





Figure 10 Variation of stress dynamic stress intensity factors of two collinear cracks for different crack distances: (a) C-F-C-F case and C-F-F-F case.



Figure 11 Two parallel off-center cracks in functionally graded piezoelectric rectangular plane: (a) C-F-C-F case and C-F-F-F case.

Two off-center equal-length cracks which are parallel to the FGPRP edges are shown in Figures 11a and 11b. The lengths of cracks remain fixed while the crack distances from the lower boundary are changed.

The variations of dimensionless DSIFs versus dimensionless time t/t_0 are depicted in Figures 12a and 12b. For the C-F-C-F case, two peaks of the curve of dynamic stress intensity factor become greater as y_c/h rises.



Figure 12 Variation of stress dynamic stress intensity factors of two off-center cracks for different crack distances: (a) C-F-C-F case and C-F-F-F case.

5 Conclusions

In this paper, the transient response of a cracked functionally graded piezoelectric rectangular plane subjected to dynamic electromechanical loads is investigated. Fourier and Laplace transform technique is implemented to obtain a new analytical transient solution for electroelastic dislocation in the rectangular plane. The solution obtained in this study could be considered to be a Green function for the cracked problem. Hence, this solution is utilized to construct integral equations for the cracked FGPRP under different boundary conditions. The effect of boundary conditions of the rectangular plane on the DSIFs is considered and discussed. Furthermore, the formulation can be extended to analyze more complicated problems involving several parallel cracks with any patterns. The numerical results also showed that it is possible to impede the crack propagation by decreasing the FG exponent.

The authors report there are no competing interests to declare.

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Appendix A:

The unknown functions in Eq. (14) for C-F-C-F case:

$$A_{1,0} = -\frac{[\mu_0 \cosh(\mu_0 \zeta) - \gamma \sinh(\mu_0 \zeta)]b_{wz}(s)e^{\gamma \zeta}\eta}{\mu_0}, \ A_{2,0} = 0,$$
 A-1

$$B_{1,0} = -\frac{\left[\mu_0 \sinh(\mu_0 h) - \gamma \cosh(\mu_0 h)\right] \left[\mu_0 \cosh(\mu_0 \zeta) - \gamma \sinh(\mu_0 \zeta)\right] b_{wz}(s) \eta e^{\gamma \zeta}}{\mu_0(\gamma \sinh(\mu_0 h) - \mu_0 \cosh(\mu_0 h))}$$
A-2

$$B_{2,0} = -\frac{(\mu_0^2 - \gamma^2)\sinh[\mu_0(h - \zeta)]b_{wz}(s)\eta e^{\gamma\zeta}}{\mu_0(\gamma\sinh(\mu_0h) - \mu_0\cosh(\mu_0h))}$$
A-3

$$C_{1,0} = -\eta(b_{\varphi z}(s) - \alpha b_{wz}(s))$$
 $C_{2,0} = 0$, $D_{1,0} = D_{2,0} = 0$ A-4

and

$$A_{1,n} = -\frac{[\mu_n \cosh(\mu_n \zeta) - \gamma \sinh(\mu_n \zeta)]b_{wz}(s)e^{\gamma \zeta}}{\beta_n \mu_n} \sin(\beta_n \eta) \quad A_{2,n} = 0, \qquad A-5$$

$$B_{1,n} = -\frac{[\mu_n \sinh(\mu_n h) - \gamma \cosh(\mu_n h)][\mu_n \cosh(\mu_n \zeta) - \gamma \sinh(\mu_n \zeta)]b_{wz}(s)e^{\gamma\zeta}}{\beta_n \mu_n [\gamma \sinh(\mu_n h) - \mu_n \cosh(\mu_n h)]} \sin(\beta_n \eta) \quad A-6$$

$$B_{2,n} = -\frac{\lambda_n^2 \sinh[\mu_n(h-\zeta)] b_{wz}(s) e^{\gamma\zeta}}{\beta_n \mu_n[\gamma \sinh(\mu_n h) - \mu_n \cosh(\mu_n h)]} \sin(\beta_n \eta)$$
A-7

$$C_{1,n} = \frac{\left[\delta_{n} \cosh(\delta_{n} \zeta) - \gamma \sinh(\delta_{n} \zeta)\right] \left[\alpha b_{wz}(s) - b_{\varphi z}(s)\right] e^{\gamma \zeta}}{\beta_{n} \delta_{n}} \sin(\beta_{n} \eta), \quad C_{2,n} = 0 \quad A-8$$

$$D_{1,n} = \frac{\left[\delta_{n} \sinh(\delta_{n}h) - \gamma \cosh(\delta_{n}h)\right]\left[\delta_{n} \cosh(\delta_{n}\zeta) - \gamma \sinh(\delta_{n}\zeta)\right]e^{\gamma\zeta}}{\beta_{n}\delta_{n}[\gamma \sinh(\delta_{n}h) - \delta_{n} \cosh(\delta_{n}h)]}$$

$$\times \sin(\beta_{n}\eta)[\alpha b_{wz}(s) - b_{\varphi z}(s)]$$
A-9

$$D_{2,n} = \frac{\beta_n \sinh[\delta_n(h-\zeta)]e^{\gamma\zeta}[\alpha b_{wz}(s) - b_{\varphi z}(s)]}{\delta_n[\gamma \sinh(\delta_n h) - \delta_n \cosh(\delta_n h)]} \sin(\beta_n \eta)$$
A-10

Appendix B:

The unknown functions in Eq. (14) for the C-F-F-F case are:

$$A_{1,0} = -\frac{[\mu_0 \cosh(\mu_0 \zeta) - \gamma \sinh(\mu_0 \zeta)]b_{wz}(s)e^{\gamma \zeta}\eta}{\mu_0}, \quad A_{2,0} = 0$$
B-1

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$$B_{1,0} = -\frac{[\mu_0 \sinh(\mu_0 h) - \gamma \cosh(\mu_0 h)][\mu_0 \cosh(\mu_0 \zeta) - \gamma \sinh(\mu_0 \zeta)]b_{wz}(s)\eta e^{\gamma \zeta}}{[\gamma \mu_0 \sinh(\mu_0 h) - {\mu_0}^2 \cosh(\mu_0 h)]} B-2$$

$$B_{2,0} = -\frac{(\mu_0^2 - \gamma^2)\sinh[\mu_0(h - \zeta)]b_{wz}(s)\eta e^{\gamma\zeta}}{[\gamma\mu_0\sinh(\mu_0h) - \mu_0^2\cosh(\mu_0h)]} B-3$$

$$C_{1,0} = -\eta(b_{\varphi z}(s) - \alpha b_{w z}(s)), \ C_{2,0} = 0$$
 B-4

$$D_{1,0} = D_{2,0} = 0$$
 B-5

$$A_{1,n} = -\frac{\left[\mu_{n} \cosh(\mu_{n} \zeta) - \gamma \sinh(\mu_{n} \zeta)\right] b_{wz}(s) e^{\gamma \zeta}}{\beta_{n} \mu_{n}} \sin(\beta_{n} \eta), A_{2,n} = 0 \qquad B-6$$

$$B_{1,n} = -\frac{[\mu_n \sinh(\mu_n h) - \gamma \cosh(\mu_n h)][\mu_n \cosh(\mu_n \zeta) - \gamma \sinh(\mu_n \zeta)]b_{wz}(s)e^{\gamma \zeta}}{\mu_n \beta_n [\gamma \sinh(\mu_n h) - \mu_n \cosh(\mu_n h)]} \sin(\beta_n \eta) \quad B-7$$

$$B_{2,n} = -\frac{\lambda_n^2 \sinh[\mu_n(h-\zeta)]b_{wz}(s)e^{\gamma\zeta}}{\mu_n\beta_n[\gamma\sinh(\mu_nh) - \mu_n\cosh(\mu_nh)]}\sin(\beta_n\eta)$$
B-8

$$C_{1,n} = \frac{\left[\delta_{n} \cosh(\delta_{n} \zeta) - \gamma \sinh(\delta_{n} \zeta)\right] \left[\alpha b_{wz}(s) - b_{\varphi z}(s)\right] e^{\gamma \zeta}}{\beta_{n} \delta_{n}} \sin(\beta_{n} \eta), \ C_{2,n} = 0 \qquad B-9$$

$$D_{1,n} = \frac{\left[\delta_{n} \sinh(\delta_{n}h) - \gamma \cosh(\delta_{n}h)\right]\left[\delta_{n} \cosh(\delta_{n}\zeta) - \gamma \sinh(\delta_{n}\zeta)\right]e^{\gamma\zeta}}{\beta_{n}\delta_{n}[\gamma \sinh(\delta_{n}h) - \delta_{n} \cosh(\delta_{n}h)]} \sin(\beta_{n}\eta)$$

$$\times \left[\alpha b_{wz}(s) - b_{\varphi z}(s)\right]$$
B-10

$$D_{2,n} = \frac{\beta_n [\alpha b_{wz}(s) - b_{\varphi z}(s)] \sinh[\delta_n (h - \zeta)] e^{\gamma \zeta}}{\delta_n [\gamma \sinh(\delta_n h) - \delta_n \cosh(\delta_n h)]} \sin(\beta_n \eta)$$
B-11