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Stabilization of Intrusive DMD based Reduced Order Model of the Convection-diffusion Equations

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Simulation of complex and non-linear dynamical systems needs numerical algorithms which are time-consuming due to hardware limitations. Therefore, many studies were focused on developing models with high speed of computation and good relative accuracy. Reduced-order model is the method that could be an alternative approach for simulating complex dynamical systems using some appropriate data from the system responses. Usually, these models are developed based on the dominant features of the desired system. Dynamic Mode Decomposition is one of the methods for calculating these basis functions. In this study, using this method and based on the concepts of dynamical systems, a reduced-order model has been developed for Burgers equation. Also, due to the incompleteness of the related modal space which is changed to low-dimensional space in order reduction procedure, the dissipation level of the surrogate model is decreased. Therefore, by using an artificial viscosity which is called the eddy viscosity approach, the stability of the model is improved. Eventually, comparing the results obtained by the stabilized reduced order model and direct numerical simulation data, the accuracy of the model with an average error of 0.4% is proven.

Keywords: Dynamic Mode Decomposition (DMD), Reduced order model, Eddy viscosity approach, Burgers equation, Stabilization approach

1 Introduction

A system whose future behavior depends on the evolution and past behavior of the system is called a "dynamical *system*".

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Mathematical models are used to predict, control and optimize dynamical systems. Usually, to correctly describe the behavior of a dynamical system, a complex mathematical model is needed, which requires a lot of calculations and a lot of storage space. For example, the fluid flow field can be introduced as a dynamical system that undergoes random changes over time. In recent decades, obtaining a fast and accurate calculation method has always been one of the concerns of researchers in the field of computational fluid dynamics. Computational limitations, limitation of calculation accuracy and limited storage space are among the most important reasons that force researchers to find simpler models. Recent developments in the field of computing hardware have been able to largely reduce the calculation time, but the interest of researchers to investigate more details of a physical phenomenon increases the complexity of the governing equations of the problems. Therefore, the need to obtain a fast and accurate model that can model the problem (with any degree of complexity) still exists. In this research, development of a surrogate model for convection-diffusion equation is studied and with the aim of reducing computational costs. In most problems of order reduction, if the governing equations are transferred to the vector space consisting of basis vectors, the field data that demonstrated the problem dynamics can be reconstructed with a smaller number of dimensions. It is an important issue when reducing the degrees of freedom and complexities of the problem, its physical characteristics are maintained. Different methods have been used to obtain the reduced order models. For example, we can refer to the use of Taylor's expansion approximation or Arnold's method [1, 2]. Bang et al. modeled the non-linear and complex models that are often encountered in real system design and analysis by the reduced-order model [3]. Callaham et al. investigated non-linear dimensionality reduction as a tool to improve the accuracy and stability of order reduction models for advection-dominated flows [4]. The method that is used more than other methods by researchers in this field to investigate the behavior of fluids is the “proper orthogonal decomposition" method. This method is one of the most common methods to reduce the order of the problem [5]. Higham et al. have applied the proper orthogonal decomposition (POD) technique to analyze granular rheology in a laboratory scale pulsed-fluidized bed. They were able to describe the inherent dynamics and energy budget in the dominant spatio-temporal modes in addition to identifying spatial coherence [6]. Moayyedi et al. used the parametric and time-dependent reduced-order model based on proper orthogonal decomposition to simulate diffusion and convection-diffusion problems [7]. Since the examination of the field structures in the proper orthogonal decomposition method is only based on the energy content of each feature and this method does not evaluate the system dynamics, the outcome reduced order model will not be able to predict the behavior of the system with high accuracy in the long term. For this reason, in recent years, by studying fluid physics and generalizing Arnoldi's method, researchers succeeded in developing a powerful method for analyzing the dynamics of non-linear systems [8,9]. This method, which is based solely on the set of data obtained from accurate numerical solutions or experimental results and has the ability to identify the areas of the field where different dynamical behaviors occur, is first introduced with the aim of extracting dynamic information of the flow [10]. Zhao et al. studied the proper orthogonal decomposition and dynamic mode decomposition for the transverse flow in the channel [11]. In their study, the optimality of the proper orthogonal decomposition for the flow reconstruction has been proven. Rowley et al. used dynamic mode decomposition to simulate the flow of a large-scale jet [12]. Sabaghzadegan et al. studied conduction heat transfer in a solid plate based on physics-informed dynamic mode decomposition. Their simulation showed the ability of this method to model the heat transfer in an equation-based model [13]. Hu et al. also investigated the flow of a centrifugal compressor using dynamic mode decomposition [14]. Sun et al. investigated an unsteady flow on a two-bladed wind turbine using vortex simulation of dynamic mode decomposition [15]. Brunton et al. developed compressed sensing strategies for computing the dynamic mode decomposition (DMD) from heavily subsampled or compressed data [16]. Using this method, Seena investigated the flow inside a cavity and succeeded in identifying modes with self-oscillation capability [17]. Bai et al. have developed a powerful mathematical framework for compressive system identification by combining DMD with compressed sensing and DMD with control approaches [18]. Moayyedi et al. studied about the instability correction of the reduced-order model of the convection-diffusion equations based on the dynamic mode decomposition at high Reynolds numbers by using the eddy viscosity approach [19]. In another study, Moayyedi et al. investigated the effect of the eddy viscosity closure in calibration of the DMD based Reduced-order model to predict the long-term behavior of convection-diffusion equations. The results showed the high accuracy and speed of the surrogate model for modeling convection-diffusion equations [20]. Schmid presented a paper to review the standard DMD, the core algorithm, as a classical matrix decomposition/factorization method, combined with a sparsity promoting optimization to extract modal amplitudes. He gave three examples that show the flexibility of dynamic mode decomposition as an analysis tool for simple to complex flow configurations [21]. Shady et al. have presented a new design-based dynamic mode decomposition framework named SketchyDMD, which is based on capturing both the range and corange of the input data. This algorithm provides fresh information about the system's singular values, eventually leading to improved approximation of the DMD spectrum. Moreover, SketchyDMD includes free parameters that can be tuned based on the given memory constraint [22]. Krake et al. used the combination of frequency-based constraints in the dynamic mode decomposition to obtain an adaptive decomposition. This leads to a constrained dynamic mode decomposition that directly addresses the problem of facilitating user interaction with the main mechanisms [23]. Baddoo et al. showed how to integrate physical principles with dynamic mode decomposition. They were able to focus on five fundamental physical principles—conservation, self-adjointness, localization, causality and shift-equivariance and derive several closed form solutions and efficient algorithms for the corresponding piDMD optimizations [24].

In this paper, using an intrusive reduced order model (combination of a data-driven method (DMD) and the related governing equation) the time evolution of the problem is modeled. Although DMD is a data-driven approach, for some problems in which the behavior of the dynamical system is very nonlinear and complex, using a physical constraint such as the governing equations, this strategy can lead to the development of a more accurate model. Finally, the outcome model has been reconstructed with less complexity and few dimensions. One of the important challenges in the development of reduced-order models is their stability under the influence of variations of important parameters such as the Reynolds number, especially at its high values. This is due to the reduction of dissipation effects with respect to the decreasing effect of the viscous term and also the changes in the reduced order model behavior resulting from the removing some of the modes. This issue has reduced the stability of the dynamical system, and its behavior may tend toward divergence. For this purpose, in this research, various methods have been used to correct the instability of the intrusive reduce order model. In most of the methods, the concept of artificial viscosity is based on the eddy viscosity concept, which is calculated in different ways. The values of eddy viscosity change according to the number of modes or the weight of modes. The outcomes from the reduced-order model compared with the exact solution data show the good accuracy of this method.

2 Viscous Burgers equation

The Navier-Stokes equation, which is like a non-linear dynamical system, due to the presence of the non-linear term in this equation, the concept of turbulence in fluid and the velocity fluctuations, can be described mathematically by this term. Burgers equation is a differential equation obtained by simplifying the Navier-Stokes equations by eliminating the pressure term.

It is clear the non-linear term in this equation is the representative term of turbulent-like behaviors, like the Navier-Stokes equations. Since Burgers equation has a non-linear behavior like Navier-Stokes equations, the methods used for Navier-Stokes equations can usually be used for this equation as well. Therefore, using burgers equation as the test field will have a suitable answer instead of the Navier-Stokes equation. This equation in dimensionless form is as:

$$\frac{\partial u}{\partial t} + (u.\nabla)u = \frac{1}{Re}\nabla^2 u \tag{1}$$

3 Direct numerical simulation

To generate the required data for constructing surrogate model, the numerical solution of the viscous Burgers equation has been used. For this purpose, to calc ulate the conservative form of the non-linear term used first order upwind method as follows:

$$u\frac{\partial u}{\partial x} = \begin{cases} u_i\dfrac{u_i - u_{i-1}}{\Delta x} & if\ (u_i > 0) \\ u_i\dfrac{u_{i+1} - u_i}{\Delta x} & if\ (u_i < 0) \end{cases} \tag{2}$$

and for the diffusion term, the second-order central difference method has been used:

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1} - 2u_i - u_{i-1}}{\Delta x^2} \tag{3}$$

Also, for time integration, the fourth order Rung-Kutta method is used [25]. The outcome numerical code has been validated by the exact solution of the one-dimensional Burgers equation [7].

4 Validation of numerical solution

In this section, the results obtained by the numerical solution of one-dimensional viscous Burgers equation have been compared with related exact solution of this equation. To do this, the results of the numerical simulation of the equation governing are compared by the data obtained from the following equation which is obtained by the exact solution of Burgers equation [17]:

$$u(x,t) = \frac{4\pi}{Re}\frac{\sum_{i=1}^{\infty} exo(\frac{-n^2\pi^2 t}{Re})I_n(\frac{Re}{2\pi})n\, sin(n\pi x)}{I_0(\frac{Re}{2\pi}) + 2\sum_{i=1}^{\infty} exp(\frac{-n^2\pi^2 t}{Re})I_n(\frac{Re}{2\pi})n\, cos(n\pi x)} \tag{4}$$

in which the initial and boundary conditions are assumed as follows:

$$\begin{gathered} u(x,0) = sin(\pi x) \\ u(0,t) = u(1,t) = 0 \end{gathered} \tag{5}$$

Considering these conditions, Burgers equation has been solved with time step of 0.001 over specific time.

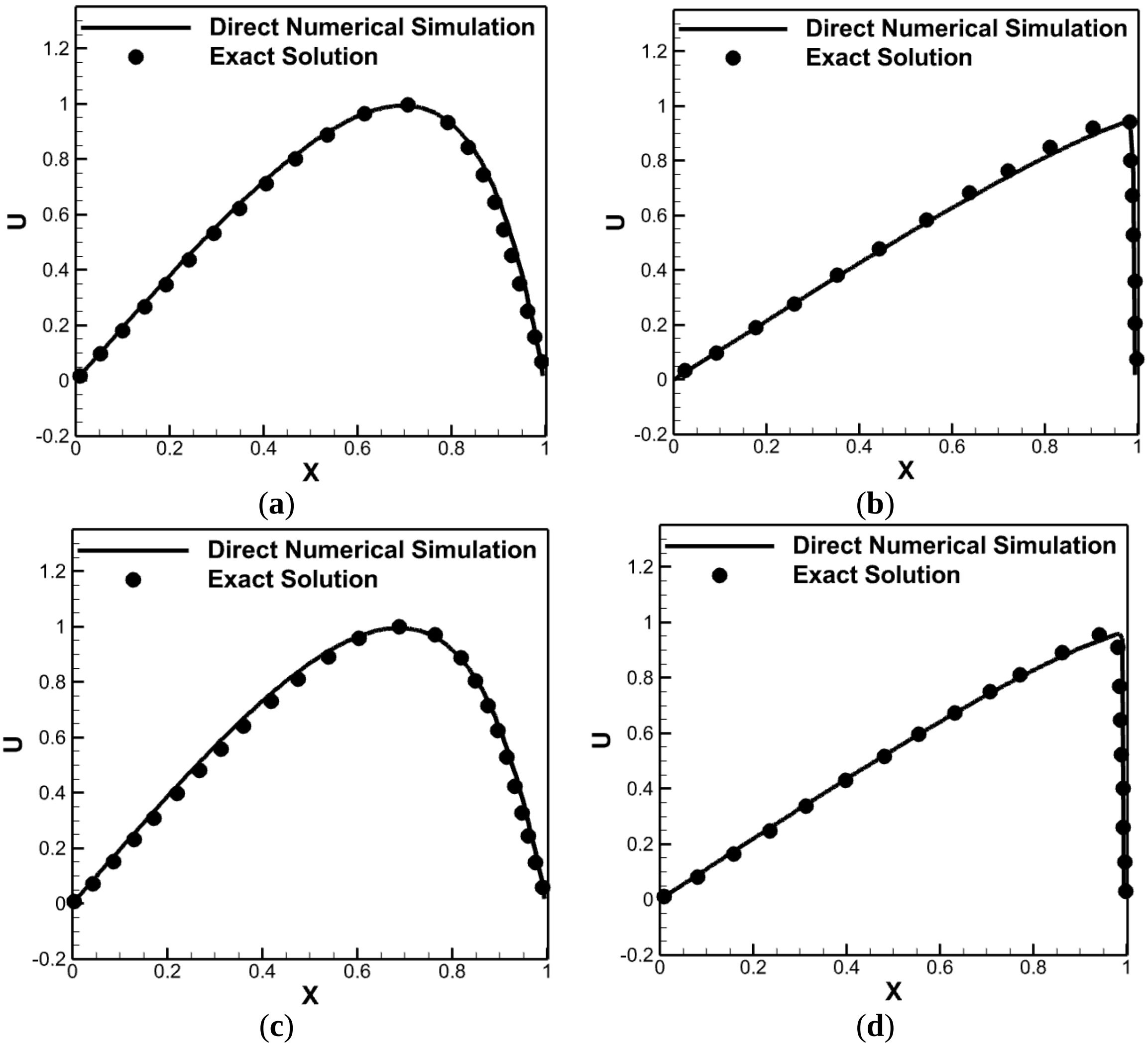


Figure 1 Comparison between the exact solution and the numerical solution data of Burgers equation at Reynolds number of 1000 at (**a**) t=0.2 (**b**) t=0.6 and Reynolds number of 10000 at (**c**) 2 t=0.2 and (**d**) t=0.6

To check the accuracy of simulation, in Fig. (1), the results of the numerical solution and the exact solution of Burgers equation at Reynolds numbers of 1000 and 10000 at t=0.2 and t=0.6 are compared. This comparison indicates the appropriate accuracy of the numerical solution to be used as ensemble data of the reduced order model.

5 Dynamic mode decomposition

The dynamic mode decomposition was introduced for the first time in a conference in 2008 by Schmid and Sesterhan and was highly welcomed [10]. Only one year later, the first archival article containing dynamic mode decomposition was published by Rowely et al. [27]. Schmid followed this with his archival article [28]. Dynamic mode decomposition is a dimensionality reduction method to decompose large-dimensional fluid flow data into a coherent spatio-temporal dominant structure. This method uses an approach like iterative methods for computing linear eigenvalues or other linear algebra problems. In this method, the goal is to use a formulation based only on the input data, which is obtained through numerical solution or experimental results. In fact, a model-based approach to dynamic information extraction has been avoided and instead focused on a data-based approach. Therefore, this method produces a set of modes along with a linear evolution model. This powerful method, which is based on data, is introduced as a method to separate the important information of the desired dynamical system (such as fluid flow) to determine the dominant features that make the dynamic behavior of the problem. Since the dynamic mode decomposition method was introduced, it has been widely used in various fields including epidemiology [29], neurosciences [30], robotics [31], video processing [32] and financial trading [33]. In fact, this method can investigate and analyze the structures of complex and non-linear systems and it is possible to identify areas of the field with different dynamic behaviors. One of the advantages of using dynamic mode decomposition is not needing to know about the governing equation of the system for some problems. Also, if the input data has been measured in the form of experimental data, it is necessary to process it to eliminate measurement errors caused by human error. In the first step, a set of data or a data matrix is needed for the dynamic mode decomposition.

In other words, it starts with a general description of the flow fields collected by sampling from direct numerical simulation or experimental data. The data that flows as a sequence of instantaneous fields will be arranged as vectors v_i . These vectors are arranged with a fixed time step relative to each other, so the system will be arranged in the form of a matrix with M rows and N columns according to Eq. (6):

$$V_1^N = [v_1, v_2, v_3, \ldots, v_N] \tag{6}$$

In the above relation, V_i it represents the instantaneous field. Now, the last index and the first index are removed from this matrix, respectively:

$$\begin{gathered} V_1^{N-1} = [v_1, v_2, v_3, \ldots, v_{N-1}] \\ V_2^N = [v_2, v_3, v_4, \ldots, v_N] \end{gathered} \tag{7}$$

Using a linear mapping, the following relationship will be established between the instantaneous field v_i , (the first set) and the instantaneous field v_{i+1} , (second set) as follow:

$$\begin{gathered} v_{i+1} = Av_i \\ V_2^N = AV_1^{N-1} \end{gathered} \tag{8}$$

The mapping matrix A , is a matrix that contains the information of the gradual evolution of the system [27]. To establish the Eq. (8), the expression should not rely on the matrix A . The non-square matrix V_1^{N-1} is decomposed as follows using the singular value decomposition method:

$$V_1^{N-1} = U\Sigma W^T \tag{9}$$

as a result:

$$V_2^N = AU\Sigma W^T \tag{10}$$

By multiplying U^T and $W S^T$ on both sides of Eq. (10), the following equation is obtained:

$$S = U^T AU\Sigma W^T W\Sigma^{-1} = U^T V_2^N W\Sigma^{-1} \tag{11}$$

The matrix S is similar to the matrix A , so we can say that the eigenvalues of the matrix S will be equal to A .

$$Sy_i = \mu_i y_i \quad i = 1, \ldots, M \tag{12}$$

where, y_i represent the eigenvector and μ_i eigenvalues of the matrix S. If both sides of the equation are multiplied by U , the same matrix U^TU will be placed on the left side of the equation.

$$USU^TUy_i = U\mu_i y_i \tag{13}$$

Therefore, the following equations will be obtained:

$$USU^T = A \tag{14}$$

$$AUy_i = \mu_i Uy \tag{15}$$

Uy_i are the eigenvectors of the matrix A and the eigenvalues of both matrices S and A are equal to μ_i . Therefore, the spatial modes obtained from the dynamic mode decomposition are as:

$$\phi_i = Uy_i \tag{16}$$

$$A\phi_i = \mu_i \phi_i \tag{17}$$

To predict the behavior of the system in the long term and reconstruct the set of initial data, a linear expansion is approximated by calculating the temporal coefficients of each mode. This is done by projecting each dynamic modes along the first snapshots according to the following relationship:

$$a_i = \varphi_{ij}^T v_{1_j} \tag{18}$$

In the above, j^T , relation is the inverse of the matrix of dynamic modes. To reconstruct each snapshot, the following equation can be used:

$$V_k = \sum_{i}^{M} a_i \varphi_{ij} \mu_i^{k-1} \tag{19}$$

For the development of the reduced order model based on dynamic mode decomposition, the concept of modes and their total energy is important. In the method used in this study, unlike the proper orthogonal decomposition, the selected modes for the development of reduced order models do not sort by its energy level. Also, the dynamic modes are not orthogonal. Therefore, by using the norm of each mode according to the Eq. (20), the relative energy of the modes can be obtained, and it is not assigned only to that mode.
However, through this relationship and the relative energy value of each mode, energetic modes can also be identified.

$$N = |a_i \varphi_{ij}| \tag{20}$$

According to Eqs. (19) and (20), the following relation can be obtained:

$$N_k = |a_i \varphi_{ij}||\mu_i|^{k-1} \tag{21}$$

In the above, due to the presence of variables m_i , the stability of the modes can be checked. To determine the importance and impact of each mode, their initial and final norm should be checked.

6 Intrusive reduced order model based on dynamic mode decomposition

Reduced-order modeling is a technique that can reduce computational complexity or computer storage requirements. This can simplify analysis, control, and design with surrogate lower-order models. To develop the reduced order model, the any quantity is written as the sum of the time averaged and fluctuation parts:

$$u(x,t) = \bar{u}(x) + \acute{u}(x,t) \tag{22}$$

The fluctuation part may be rewritten based on the Galerkin approximation as follows:

$$\acute{u}(x,t) = \sum_{i=1}^{N} a^i(t)\phi_i(x) \tag{23}$$

where, f_i is arbitrary basis function and a^i are modal coefficients and their values should be chosen in a way to satisfy the differential equation with a good approximation. By implementing the relations of the time averaged and the fluctuation parts into burgers equation, the following relation is obtained:

$$\begin{aligned}\phi_i(x) \times \sum_{1}^{N} \frac{d}{dt}\left(a^i(t)\right) + (\bar{u} \times a^i(t) \times \sum_{1}^{N} \nabla\phi_i(x) + \bar{u} \times \sum_{1}^{N} \nabla\bar{u} + \ a^i(t) \\ \times \sum_{1}^{N} \nabla(\bar{u}, \phi_i(x)) + a^i(t) \times a^j(t) \times \sum_{i=j=1}^{N} \nabla(\phi_i(x), \phi_j(x))) \\ = \frac{1}{Re}(\nabla^2\bar{u} + a^i(t) \times \sum_{1}^{N} \nabla^2\phi_i(x))\end{aligned} \tag{24}$$

For POD modes, which are orthogonal, the coefficient of the transient term in reduced-order model is simplified to the identity matrix. But in the dynamic mode decomposition, as mentioned earlier, the modes do not necessarily have orthogonality, so a quasi-orthogonal addition basis function is necessary to multiply the matrix on both sides of the governing equation. This term is defined as follows:

$$\phi^{ad} = ((\phi_k^H(\vec{x}), \phi_i(\vec{x}))^{-1} \tag{25}$$

In the above relation, f_k^H is the conjugate of the transposed modes. By multiplying both sides of the Eq. (24) by this matrix, the following equation will be obtained:

$$
\begin{aligned}
&(\phi_k^H(x),\phi_i(x)) \times \sum_1^N \frac{d}{dt}(a^i(t)) + (\phi_k^H(x),\bar{u}\cdot\nabla\phi_i(x)) \times a^i(t) \; - \; (\phi_k^H(x),\bar{u}\cdot\nabla\bar{u}) \\
&\qquad - \; (\phi_k^H(x),\phi_i(x)\cdot\nabla\bar{u}) + (\phi_k(x),\phi_i(x)\cdot\nabla\phi_j(x)) \times a^i(t) \times a^j(t) \\
&\qquad = \frac{1}{Re}(\phi_k^H(x),\nabla^2\bar{u}) + \frac{1}{Re}(\phi_k^H(x),\nabla^2\phi_i(x)) \times a^i(t)
\end{aligned}
\tag{26}
$$

Finally, the equation of the reduced-order model will be obtained as a simple first-order differential equation for the time-dependent modal coefficients:

$$
\frac{da^k}{dt} = \tilde{A}_{kij} \times a^i(t) + \tilde{B}_{ki} \times a^i(t) + \tilde{C}_k \tag{27}
$$

The outcome model is a dynamical system that can be used to calculate temporal modal coefficients over time. The coefficients in Eq. (27) are calculated as follows:

$$
\tilde{A}_{kij} = A_{ij}^k \times \phi^{ad} \qquad\qquad \tilde{B}_{ki} = B_i^k \times \phi^{ad} \qquad\qquad \tilde{C}_k = B^k \times \phi^{ad}
$$

So that:

$$
\begin{aligned}
&A_{ij}^k = (\phi_i(x)\cdot\nabla\phi_j(x),\phi_k^H(x)) \times \phi^{ad} \\
&B_i^k = (\bar{U}\cdot\nabla\phi_i(x),\phi_k^H(x)) + (\phi_i(x)\cdot\nabla\bar{U},\phi_k^H(x)) \times \phi^{ad} \; - \; \frac{1}{Re}(\nabla^2\phi_i(x),\phi_k^H(x)) \times \phi^{ad} \\
&C^k = (\bar{U}\cdot\nabla\bar{U},\phi_k^H(x)) \times \phi^{ad} \; - \; \frac{1}{Re}(\nabla^2\bar{U},\phi_k^H(x)) \times \phi^{ad}
\end{aligned}
$$

7 Sources of instabilities in hybrid data-driven physics informed models

Reduced-order modeling of dynamic systems with nonlinear and relatively complex physics can lead to unacceptable or non-physical results under the influence of various factors and does not have high accuracy in predicting the temporal variations of the dynamical systems. In this case, the behavior of the model can lead to incorrect results in a short or long period of time. This divergence from the correct results is not only specific to the models based on the dynamic mode decomposition but is also observed in the reduced order models obtained from the proper orthogonal decomposition. Therefore, it can be said that the system of ordinary differential equations obtained from the Galerkin projection of governing equation which is used to model the behavior of the system under the variation of the important parameters, may be unstable. In the present study, if the system response is calculated at small time steps and low Reynolds numbers, the results obtained from the direct numerical simulation and the reduced-order model are completely similar and an accurate prediction of the behavior of the problem is obtained. One of the reasons for this issue can be the dominance of the diffusion term in the governing equation and the effect of increasing its dissipation effects. In the development of the reduced order model, the order reduction manner which is based on removing the effects of some modes, has been used. This issue is also an important factor in reducing the sufficient dissipation and the stability of the response of the reduced order model. But with the increase of Reynolds number, the viscous term in the governing equation has less effect and like the turbulent flow, the dissipation term in the governing equation will be reduced. Therefore, the dissipation

required to guarantee the stability of the reduced order dynamical system is reduced and the results may be deviated from the accurate behavior. Dynamic mode decomposition and some of the similar methods are usually addressed as a data-driven method. The input data for these methods are obtained from either experimental tests or numerical models on a real system.
In the experimental data acquisition, there may be errors related to equipment and measurement. Therefore, the surrogate model developed based on these data may contain some inherent errors which lead to an unstable model that cannot predict the behavior of the desired dynamical system accurately. For numerical model data, some of the reasons such as numerical algorithm accuracy, grid resolution, and time step size (for unsteady problems) may be sources of generation of low-resolution dataset. Therefore, the data-driven surrogate model cannot predict the time evolution of the system with high accuracy.

8 Model stabilization approaches

As mentioned in the previous section, due to the increase of the Reynolds number and the subsequent reduction of the necessary dissipations, as well as considering a limited number of modes, instability will be increased in the behavior of the reduced order model and some results will not have the right answer over time. Therefore, to remove the instability and compensate for the lost dissipation, an artificial viscosity term is used in the model to reach stability again. Also, this term can be used as a substitute for the effect of modes removed in the order reduction process and like the turbulent flow simulation approach for modeling homogeneous structures that have a lower energy level. The desired coefficient to stabilize the reduced order model is variable for different Reynolds numbers, but in general, in this research its value is added to the constant and linear coefficient of the standard reduced order model as follows.

$$
\begin{aligned}
b_k^2 &= \langle \nu_e \psi_k \nabla^2 \bar{u}, \varphi_k \rangle \\
L_{ik}^1 &= \langle \nu_e \psi_k \nabla^2 \varphi_i, \varphi_k \rangle
\end{aligned}
\tag{28}
$$

In this model, the coefficient of the diffusion operator is constant in space and time, but it can also depend on the modes.

In Eq. (26) the eddy viscosity, n_e , is assumed to be a constant for different Reynolds values and does not depend on the different modes. y_k , represents the eddy viscosity coefficient that can depend on each mode. In the following, various kernel closure models of obtaining the eddy viscosity coefficient and their effect on stabilizing the dynamical system will be discussed. In general, seven different approaches have been investigated. For the development of the reduced order model based on dynamic mode decomposition, the modes characteristics and their total energy is important, but it is necessary to state that because the modes obtained from DMD don't have energy level superiority over each other. So, the modes don't have orthogonal properties and are not independent of each other. Therefore, in this research unlike the proper orthogonal decomposition where the eddy viscosity coefficient will be calculated according to the energy level of each mode, the summation of the eddy viscosity coefficient related to each mode is used. The first and simplest closure model in this category is Heisenberg's method [34] that is called DMD-ROM-H model.

$$
\psi_k^{DMD-ROM-H} = 1 \tag{29}
$$

Then, the DMD-ROM-H method was developed by Rempfer [35] and presented as the DMD-ROM-R method and obtained in the form of Eq. (30) that uses the following linear viscosity kernel:

$$\psi_k^{DMD-ROM-R} = \frac{k}{R} \qquad k = 1,2,3,\ldots,R \tag{30}$$

where R , is the number of assumed modes in DMD method. In this study, 2 new stabilization closure models of the DMD-ROM-R method were introduced by san [36], which are examined here. The first closure model uses a quadratic term as a viscosity kernel and is called DMD-ROM-RQ:

$$\psi_k^{DMD-ROM-RQ} = \left(\frac{k}{R}\right)^2 \tag{31}$$

The next approach, denoted as DMD-ROM-RS, uses the square-root of viscosity kernel:

$$\psi_k^{DMD-ROM-RS} = \left(\frac{k}{R}\right)^{\frac{1}{2}} \tag{32}$$

The significant differences between the two DMD-ROM-RQ and DMD-ROM-RS are quite clear. Another closure model described by Tadmor [37] is called DMD-ROM-T and will be in the following form:

$$\psi_k^{DMD-ROM-T} = \begin{cases} 0 & k \leq M \\ 1 & k > M \end{cases} \tag{33}$$

In this equation, M is a free parameter and must always have a value less than the number of modes. Sirisup and Karniadakis also developed the concept of spectral viscosity and presented a model called DMD-ROM-MK for the stability of the dynamical system resulting from the order reduction method, which will be in the following form [38]:

$$\psi_k^{DMD-ROM-MK} = \begin{cases} e^{-(k-R)^2 / (k-M)^2} & k \leq M \\ 0 & k > M \end{cases} \tag{34}$$

and finally, the last closure model uses the following equation as the eddy viscosity coefficient, which was presented by Chollet [39] and Lesieur [40]. This model is called DMD-ROM-CL and it will be in this form:

$$\psi_k^{DMD-ROM-CL} = k_0^{-3/2}\left[k_1 + k_2 e^{\frac{-k_3/k}{R}}\right] \tag{35}$$

In this equation, the constant values are as follows:

$$k_0 = 1.1135 \qquad k_1 = 0.441 \qquad k_2 = 15.2 \qquad k_3 = 3.03$$

9 Results

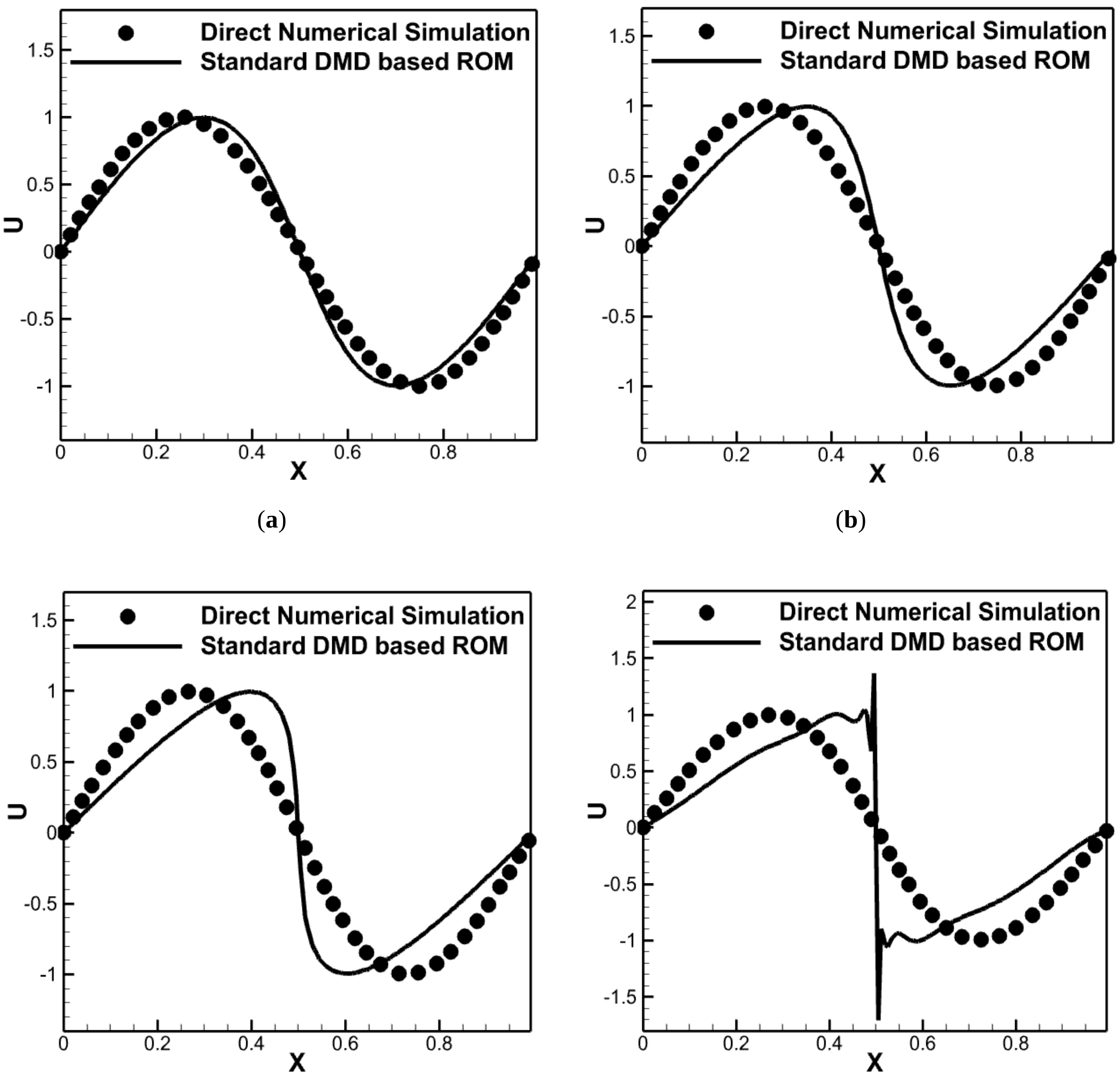
In this section, the results obtained from DMD-ROM closure models which mentioned in the previous section, are examined. Reduced order model based on DMD for Burgers equation at two Reynolds numbers of 1000 and 5000 has been used.

The outcomes have been compared with the results obtained from the direct numerical simulation method to verify the accuracy of the surrogate model. Therefore, from DNS with a time step of 0.001, a matrix contains 100 members with the equal time interval is obtained. This dataset used for training surrogate model as an input data. For this problem, the initial and boundary conditions used as follow:

$$u(x,0) = sin(2\pi x) \tag{36}$$

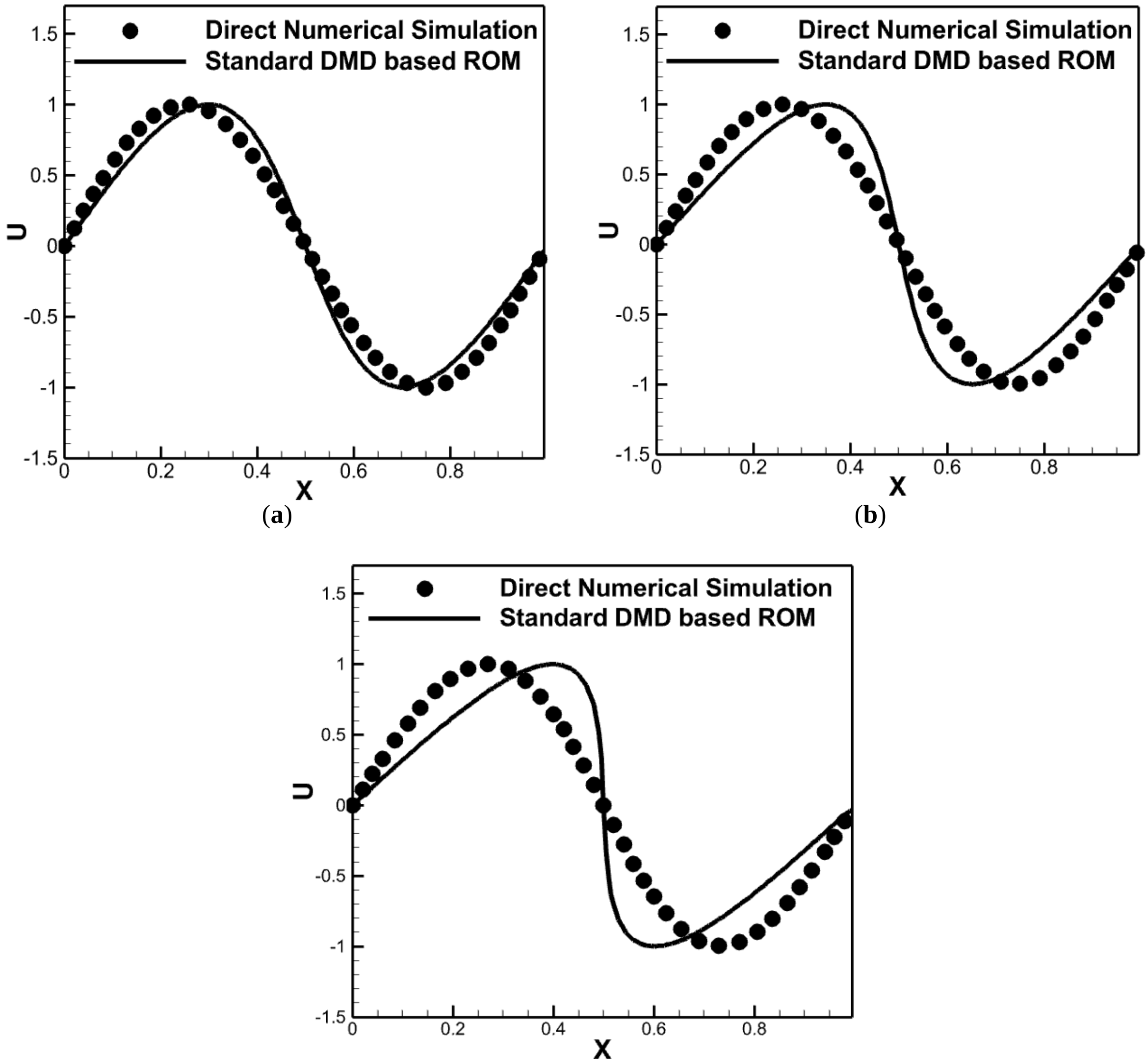
$$u(0,t) = u(1,t) = 0 \tag{37}$$

After solving the eigenvalues problem, the DMD spatial and temporal modes will be calculated. In Fig. (2), the results obtained from reduced order model and DNS data are compared for Reynolds number of 1000 at the time steps of 0.1, 0.2, 0.3 and 0.4. According to this figure, the accuracy of ROM has decreased over time which means that the model cannot predict the whole dynamics well.



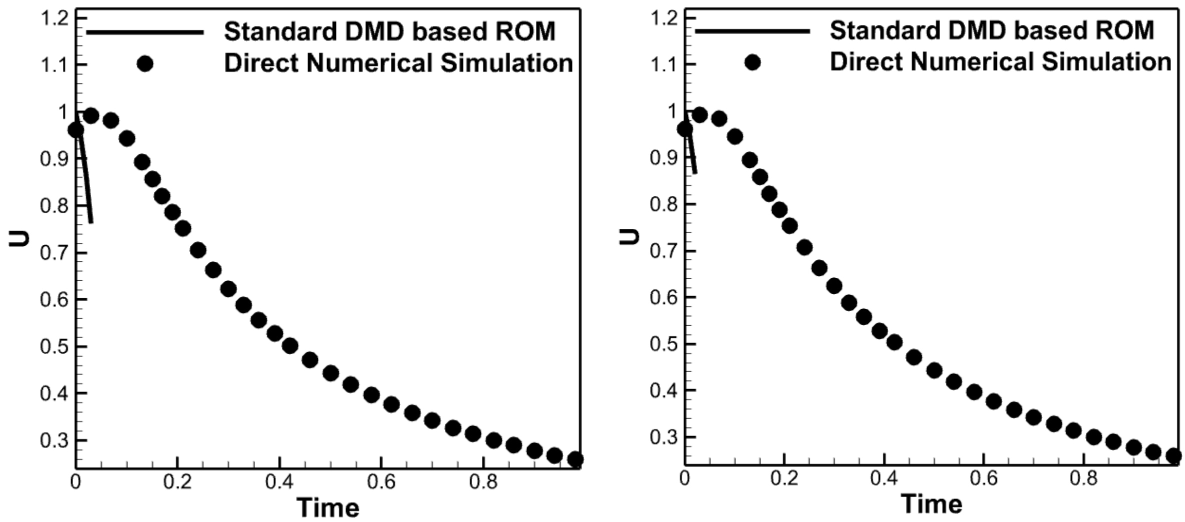
(**a**) (**b**) (**c**) (**d**)

Figure 2 Comparison between results obtained by Standard Reduced order model and direct numerical simulation of Burgers equation for Reynolds number of 1000 at (**a**) t=0.1, (**b**) t=0.2, (**c**) t=0.3 and (**d**) t=0.4



(a) (b) (c)

Figure 3 Comparison between results obtained by standard Reduced order model and direct numerical simulation of Burgers equation for Reynolds number of 5000 at (**a**) t=0.1, (**b**) t=0.2 and (**c**) t=0.3



(a) (b)

Figure 4 Comparison between results obtained by standard Reduced order model and direct numerical simulation of Burgers equation on x=0.25 for Reynolds numbers of (**a**) 1000, (**b**) 5000

Deviation of reduced order model outcomes from the exact solution is increased for Reynolds number of 5000 which is shown in Fig. (3) at time steps of 0.1, 0.2 and 0.3. So, it is clear that the accuracy of ROM obviously has decreased over time. Since Reynolds number increases, the nonlinear behavior of the problem becomes dominant, and the deviation of ROM increases over time. In other words, a discontinuity has been created at large time steps according to the physical conditions assumed, which indicates the reduction of the effect of the dissipative term in the governing equation. This behavior is more evident with the increase of the Reynolds number, because in Fig. (3) the results obtained by ROM can only be seen up to time step of 0.3 and the surrogate model gives unknown and non-physical results for the larger time steps. To investigate the time-dependent behavior of the model in predicting of the Burgers equation response, the results of ROM and DNS were calculated at a specific point (x = 0.3) and for two Reynolds numbers which are shown in Fig. (4) As shown, the results deviated and lead to meaningless values that cannot be shown in the figures. Based on these experiments, developing a stabilized reduced order model for this type of equation and in similar physical conditions, which are previously explained, seems important.

To study the accuracy of different stabilization methods, the results of the reduced order model for Reynolds numbers of 1000 and 5000 at the last time step have been compared with the DNS data. As mentioned in the methodology of the solution and development of ROM, to remove the instability of the model by using an artificial dissipation like eddy viscosity, the model has reached stability to accurate prediction at the different Reynolds numbers. In all stabilization approaches, the value of eddy viscosity is considered constant according to each method. The value of eddy viscosity and the number of modes for each closure model are considered like to the data listed in Table (1).

Fig. (5) shows the results obtained by DMD-ROM-H closure model of Burgers equation. The results have been compared with the direct numerical simulation for two Reynolds numbers in the last time step. According to the behavior of the model, it can be said that this closure model will be able to predict the dynamics of the system at different Reynolds numbers with various physical characteristics. Comparison of the results of DMD-ROM-R closure model with DNS data of Burgers equation at two different Reynolds numbers at the last time step is shown in Fig. (6). The results show the acceptable accuracy of this closure model and confirm controlling the instability of ROM.

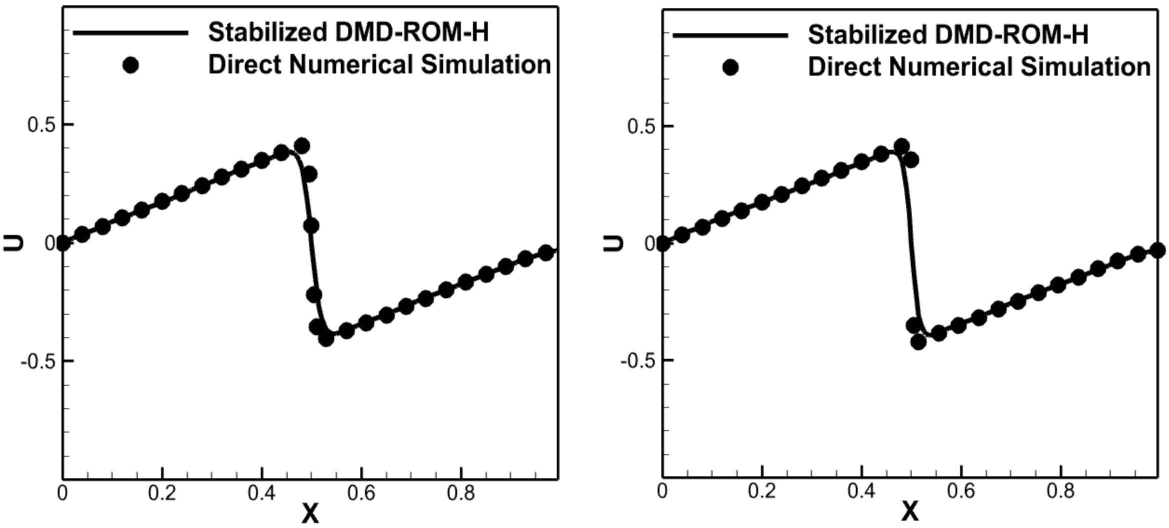
The results obtained by DMD-ROM-RQ closure model are compared with DNS data at the last time step and for two Reynolds numbers and shown in Fig. (7). As is known, by applying this closure model the problem dynamics will have a much more accurate response for both Reynolds numbers compared to the standard ROM. Likewise, Fig. (8) shows the comparison between the results of the stabilized closure model (DMD-ROM-RS) at the last time step and for two Reynolds numbers. It can be clearly seen that this closure model will be qualified as a relevant surrogate model of the governing equation.

Table1 Eddy viscosity and number of modes used for each closure model

Closure Model	Eddy Viscosity	Number of Modes
DMD-ROM-H	0.0032	10
DMD-ROM-R	0.0005	7
DMD-ROM-RQ	0.0005	7
DMD-ROM-RS	0.001	7
DMD-ROM-T	0.0012	10
DMD-ROM-MK	0.0005	7
DMD-ROM-CL	0.0012	6

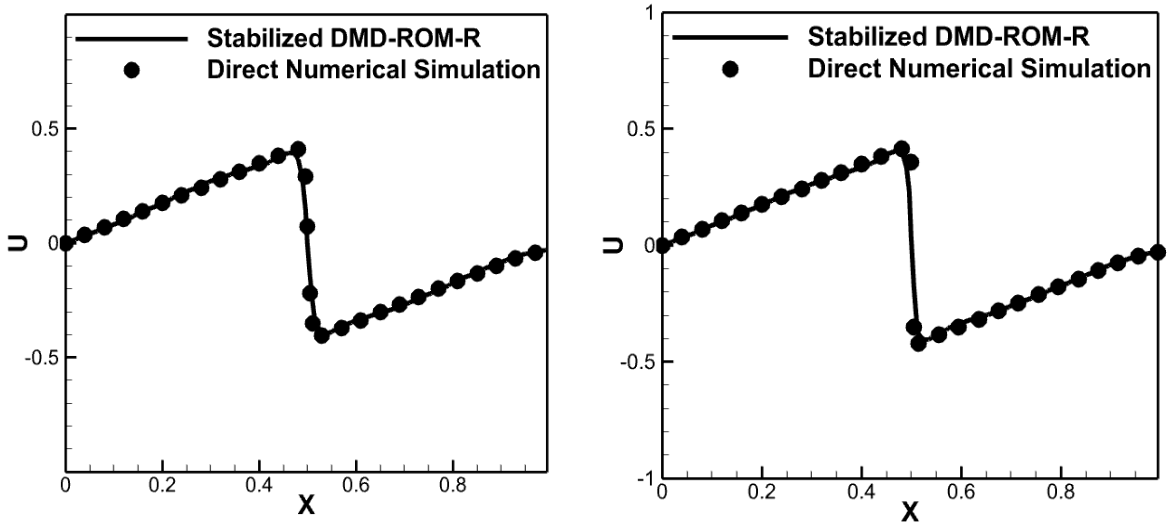
In Fig. (9), the results of DMD-ROM-T closure model compared with the relevant DNS data at the last time step for two different Reynolds numbers is shown. By using this stabilization closure model, the results obtained by ROM are well achieved and the model will be accurate enough to predict the problem dynamics. Also, Fig. (10) shows the results of the stabilized DMD-ROM-MK model at the last time step compared with DNS data. Also, by applying this closure approach, ROM has been able to predict the problem dynamics with acceptable accuracy.

The last stabilization model is DMD-ROM-CL closure model, and its outcomes are shown in Fig. (11) compared with DNS results for two different Reynolds numbers and at the last time step. According to this figure, the stabilization term added to the governing equation could remove the instability of the reduced order model, and the final model has reached stability using this closure model.



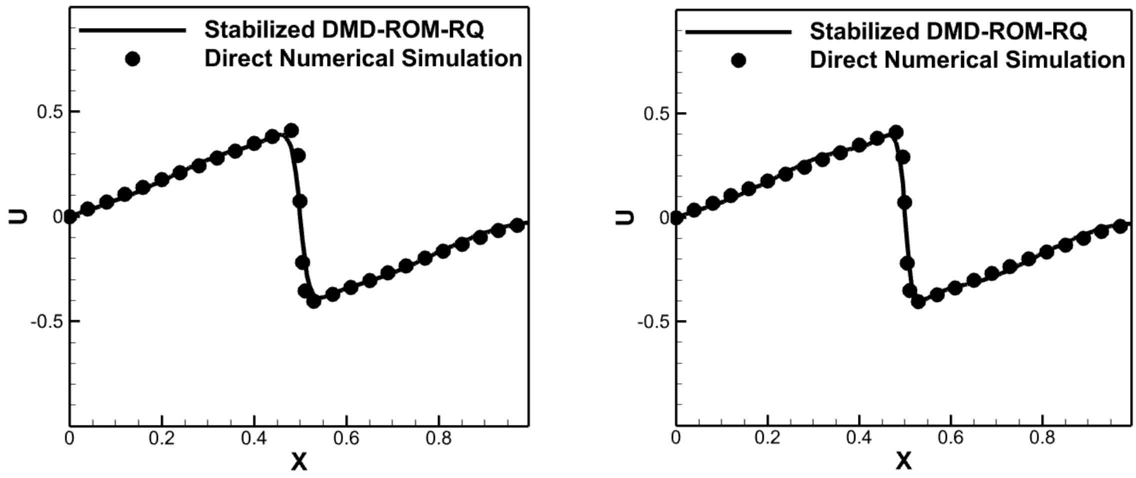
(a) (b)

Figure 5 Comparison between the prediction of DMD-ROM-H closure model at the last time step and direct numerical simulation data for Reynolds number of (**a**) 1000 and (**b**) 5000



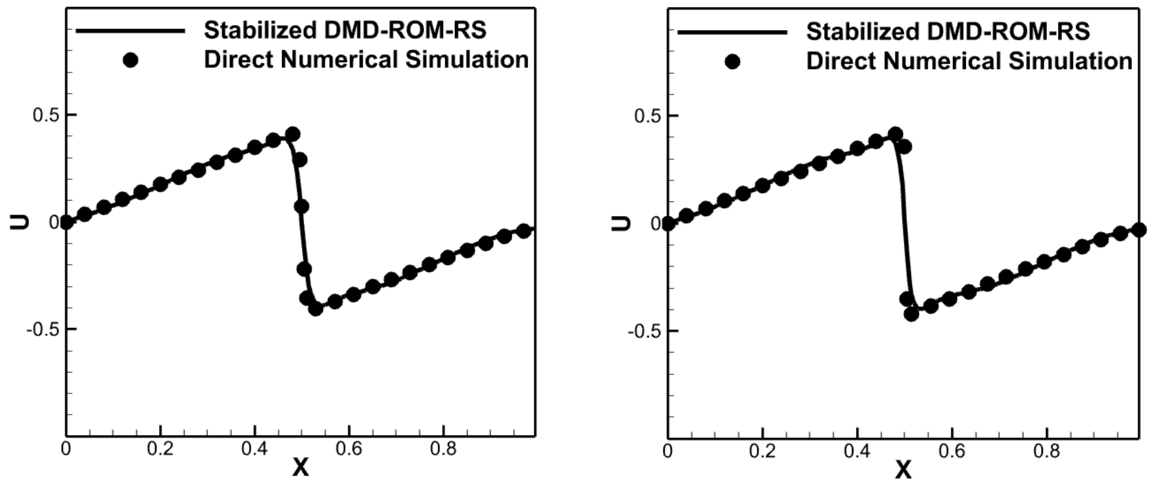
(a) (b)

Figure 6 Comparison between the prediction of DMD-ROM-R closure model at the last time step and direct numerical simulation data for Reynolds number of (**a**) 1000 and (**b**) 5000



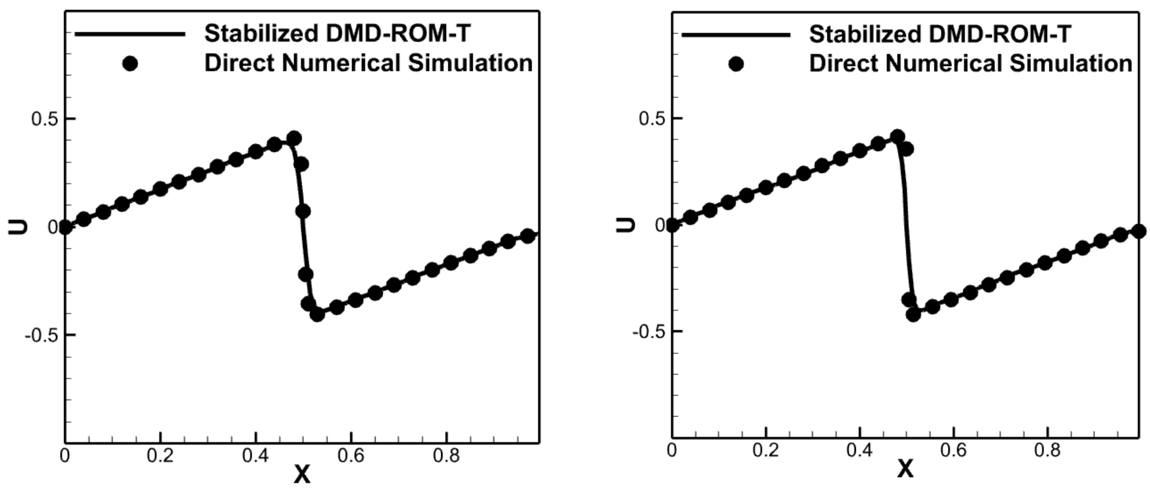
(**a**) (**b**)

Figure 7 Comparison between the Prediction of DMD-ROM-RQ closure model at the last time step and direct numerical simulation data for Reynolds number of (**a**) 1000 and (**b**) 5000



(**a**) (**b**)

Figure 8 Comparison between the Prediction of DMD-ROM-RS closure model at the last time step and direct numerical simulation data for Reynolds number of (**a**) 1000 and (**b**) 5000



(**a**) (**b**)

Figure 9 Comparison between the prediction of DMD-ROM-T closure model at the last time step and direct numerical simulation data for Reynolds number of (**a**) 1000 and (**b**) 5000

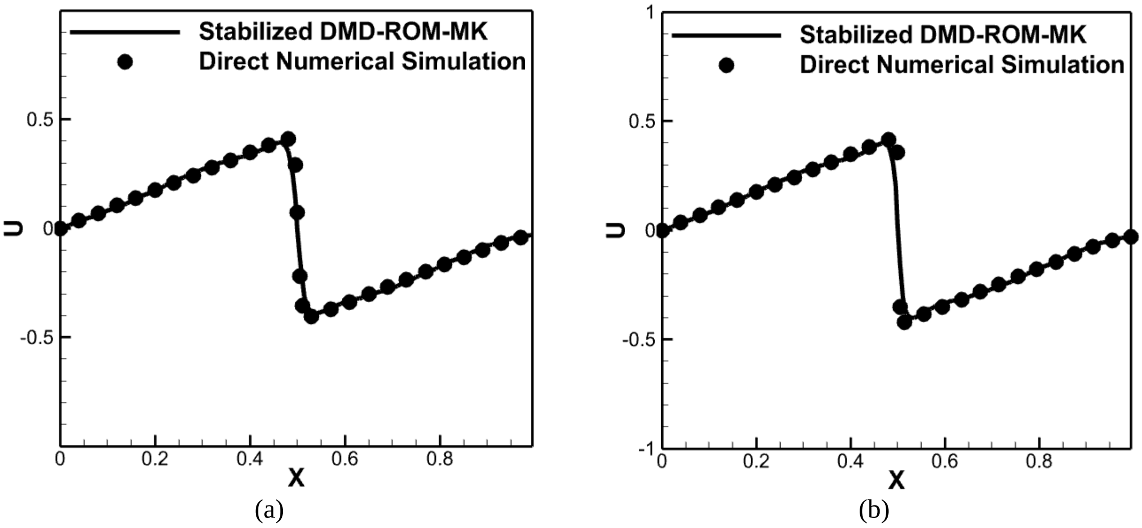


Figure 10 Comparison between the Prediction of DMD-ROM-MK closure model at the last time step and direct numerical simulation data for Reynolds number of (**a**) 1000 and (**b**) 5000

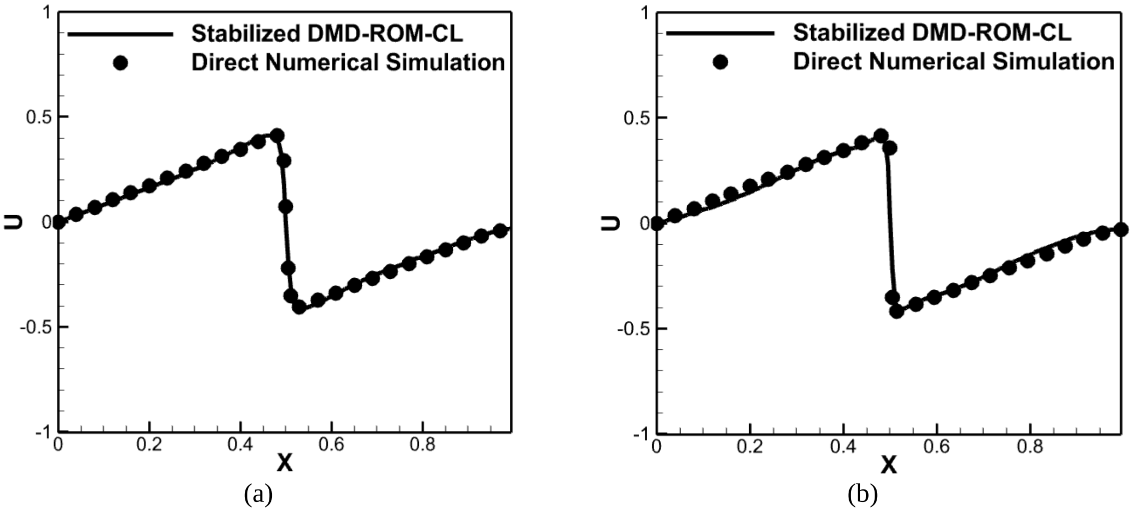


Figure 11 Comparison between the Prediction of DMD-ROM-CL closure model at the last time step and direct numerical simulation data for Reynolds number of (**a**) 1000 and (**b**) 5000

Using all stabilization models, it was found that the results obtained from the reduced-order model based on dynamic mode decomposition were able to reach the required stability. So, the effect of the artificial dissipative term applied in the governing equation instead of the dissipation reduction effect, is due to the increase of Reynolds number and the effect of removed modes in the order reduction manner. In other words, the stabilized ROM will have a more accurate prediction compared to standard ROM. To better understand and to investigate the time-dependent behavior of the stabilized ROM, the response of the problem obtained by the stabilized model of Burgers equation at a specific point ($x = 0.3$) has been computed over time. In Fig. (12), a comparison is made between the results of DNS and stabilized ROM by all seven closure models for Re =1000. Also, outcomes of the stabilized model are compared with DNS at this specific point for Reynolds number of 5000 in Fig. (13). As mentioned before, due to the increasing Reynolds number and then the reduction of the effects of the viscous term and the effective dissipation in the surrogate model. Also, the generation of a surrogate model by only a few numbers of modes will not provide right results over time and this problem can be solved by closure model. Therefore, by using these 7 mentioned closure methods, the model has reached stability again and accurately predicts the problem dynamics.

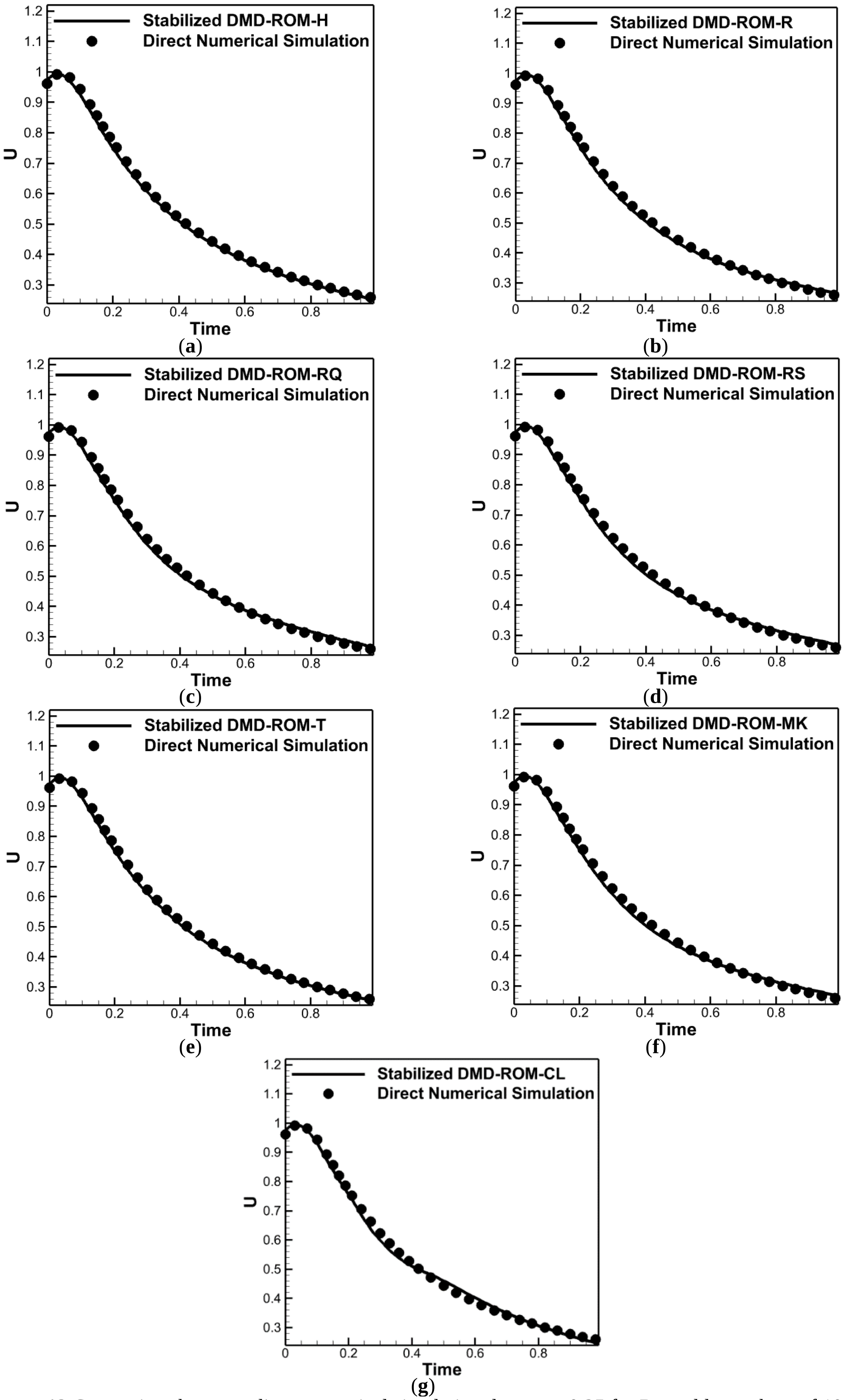


Figure 12 Comparison between direct numerical simulation data at x=0.25 for Reynolds numbers of 1000 and the prediction of closure Reduced Order Model by (**a**) H, (**b**) R, (**c**) RQ, (**d**) RS, (**e**) T, (f) MK and (**g**) CL stabilization approaches

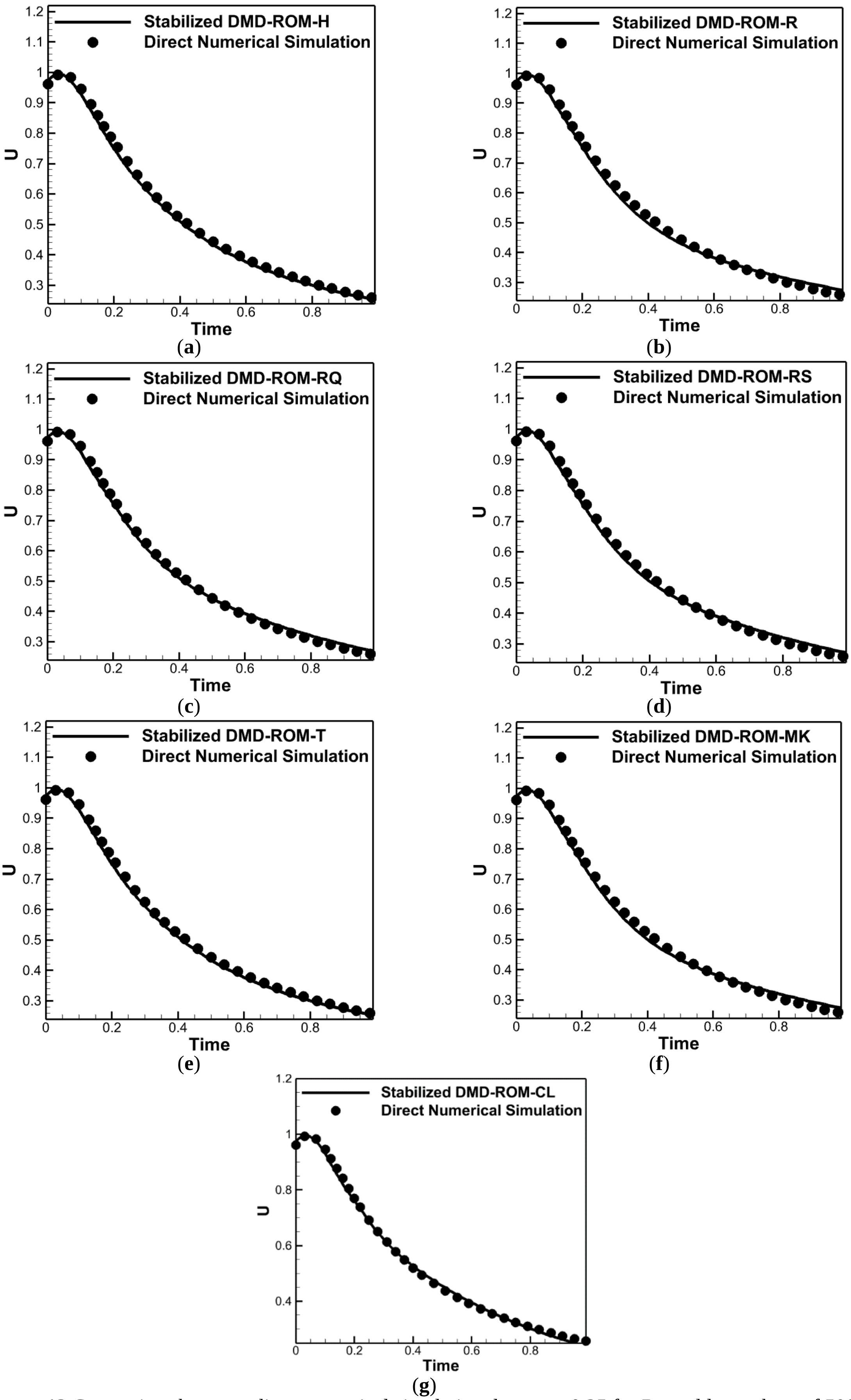


Figure 13 Comparison between direct numerical simulation data at x=0.25 for Reynolds numbers of 5000 and the Prediction of closure Reduced Order Model by (**a**) H, (**b**) R, (**c**) RQ, (d) RS, (**e**) T, (f) MK and (**g**) CL stabilization approaches

Table1 The prediction error of each closure model

	Maximum Error	
Closure Model	**Re=1000**	**Re=5000**
DMD-ROM-H	2.2×10^{-3}	4.36×10^{-3}
DMD-ROM-R	1.56×10^{-3}	9.3×10^{-3}
DMD-ROM-RQ	2.01×10^{-3}	3.93×10^{-3}
DMD-ROM-RS	1.78×10^{-3}	3.7×10^{-3}
DMD-ROM-T	1.57×10^{-3}	3.62×10^{-3}
DMD-ROM-MK	1.57×10^{-3}	3.52×10^{-3}
DMD-ROM-CL	2.91×10^{-3}	2.1×10^{-2}

It was found the accuracy of the prediction is quite enough to reconstruct the flow field in all DMD-ROM closure models. However, using the root-mean-square error method, the deviation between the prediction of the stabilized reduced order model and direct numerical simulation data has been calculated. Root-mean-square is one of the most commonly used measures for evaluating the quality of predictions and can be expressed as:

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N} \|y(i) - \hat{y}(i)\|^2}{N}} \tag{38}$$

where N is the number of data points, $y(i)$ is the i-th measurement and $\hat{y}(i)$ is the corresponding prediction. Therefore, the prediction error due to stabilized reduced order modeling compared to direct numerical simulation data using the root-mean-square error method is calculated over time and the maximum value of it at a specific time is listed in Table. (2). According to this table, the prediction error increases with increasing Reynolds number, but it is still able to model the flow field. Furthermore, DMD-ROM-CL method has the highest error rate among all methods.

Conclusions

In this study, the convection-diffusion equation, like Burgers equation, is considered to build up a data-driven equation-based reduced order framework. The reduced order model based on dynamic mode decomposition has been developed. The order reduction manner is performed based on removing some modes and considering a few numbers of them. This issue and reduction of the effects of the viscous term due to the increasing Reynolds number will cause increasing the instability of the reduced-order dynamical system. Therefore, the results of the classical reduced-order model become incorrect and diverge from exact data over time, which can be seen more clearly at the Reynolds number of 5000. To stabilize and compensate for this lost dissipation, a term called artificial dissipation is added to the model equation.

In this research, different methods of calculating the eddy viscosity coefficient have been studied. In general, seven approaches have been investigated that in all of them, the new term can compensate the effect of the modes which are removed in the order reduction manner. So, the dynamical system has reached stability by applying this term and the results of stabilized ROM through all the mentioned methods based on DMD method in different Reynolds numbers and in all time steps will be acceptable.

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