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Linear Thermoplastic Analysis of FGM Rotating Discs with Variable Thickness

This work presents thermoplastic analysis of FG rotating diskswith variable thickness and constant angular velocity. The solutions are obtained by variable material property (VMP) theory. In this theory, the domain is divided into some finite sub-domains in the radial direction, in which the properties are assumed to be constant and the form of the elastic response is used to solve elastic-plastic problems. The results are compared with the results obtained by the finite element analysis using ANSYS software. The results reveal that mentioned methods are in very good agreement in both thermoelastic and themroplastic states. Finally, the effect of various parameters including the thermal distribution and thickness profile on the stress behavior of disk are investigated.

Keywords: functionally graded material, rotating disk, thermoplastic analysis, variable material property theory.

1 Introduction

Today, functionally graded materials (FGMs) play an important role in the aerospace industry due to their high thermal and mechanical properties.The main application of FGMs is in high temperature such as automotive, aircrafts, turbine rotors, flywheels, gears etc. In these materials, the volume fraction of the two or more materials is varied steadily and nonhomogeneously as a function of position along thickness. Usually, a ceramic is used attheouter surface and a metal is used to anothersurface, which the volume fraction changes steadily. Within FGMs the different microstructural phases have differentfunctions, and the overall FGMs attain the multistructural status from their property gradation. By gradually varying the volume fraction of constituent materials, their material properties exhibit a smooth and continuous change from one surface to another [1].

Rotating discs are widely used in various applications in aerospace industries such as gas turbines, jet engines, flywheels, cars, pumps, compressors etc. Rotating discs are usually operating at high angular velocities and subjected to thermo-mechanical loadings. Recent

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studies have shown that, at the same angular velocity, the stresses developed in a rotating disk (hollow or solid) with variable thickness are much lower than those of a disk with uniform thickness [2].

Naghdabadi and HosseiniKordkheili [3] have derived a finite element formulation for the thermoelastic analysis of FG plates and shells. They assumed the power law distribution model for the composition of the constituent materials in shell thickness direction.

Bayat et al. [4] implemented a semi-analytical method to elastic analysis of functionally graded rotating disk with variable thickness (see Figure (1)). The material properties and disk thickness profile are assumed to be presented by two power-law distributions. Results revealed that, in functionally graded rotating disk with parabolic or hyperbolic convergent thickness profile, stresses and displacements are smaller than that with uniform thickness profile.

This work aims to use variable material properties theory for analyzing the stress behavior of rotating discs made of functionally graded with variable thickness and constant angular velocity under thermo-mechanical loading. For this purpose, the domain is divided into some finite sub-domains in the radial direction, in which the properties are assumed to be constant and the form of the elastic response is used to solve elastic-plastic problems. The results obtained by the VMP method are then compared with the results obtained by the finite element analysis using ANSYS software.

Finally, the effect of various parameters including the thermal distribution andthickness profile on the stress behavior of disk was investigated.

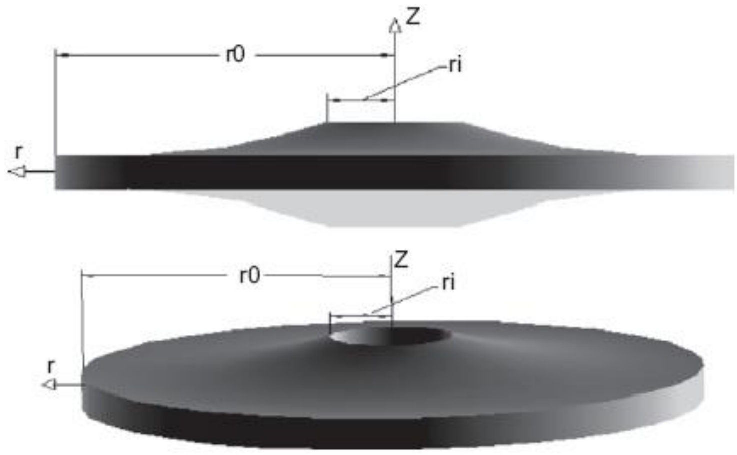

Figure 1 Configuration of a thin disk with variable thickness [4].

2 Mathematical model

2.1 Gradation relation

A functionally graded rotating disk (inner radius r_i , outer radius r_o , and angular velocity ω) made by mixing two distinct material phases, for example a metal and a ceramic, is considered with cylindrical polar coordinates *r*, *θ*, and *z* (see Figure (1)). It is assumed that, the volume fraction of metal and ceramic follow a simple power law:

$$V_{m=}\left(\frac{r-r_i}{r_o-r_i}\right)^N \quad (1)$$

$$V_c = 1 - V_m$$

Where V_C and V_m are the volume fractions ofceramic and metal, respectively the volume fraction index N dictates the material variations profile through the FGM.

So, the effective material properties of the FGM layer, P_f , such as Young's modulus, Poisson's ratio, thermal expansion coefficient, density, yield strength, and tangent modulus can then be expressed as:

$$P_f = P_m V_m + P_C V_C \tag{2}$$

Where P_m and P_C are the material properties of ceramic and metal, respectively.

2.2 The Disk Model

A functionally graded rotating disk (inner radius r_i, outer radius r_o and angular velocity ω) made by mixing two distinct material phases, for example a metal and a ceramic, is considered with cylindrical polar coordinates r, θ, and z. Its thickness profile vary radially in a form given by

$$h(r) = h_o((\frac{r}{r_o})^{-m}) \tag{3}$$

Where m and h_o are geometric parameters. m can be negative or positive and h_o is the thickness at the axis of the disk.

Different forms of thickness profile of disk with negative, zero, and positive values of m are shown in Figure (2). It can be seen that the profile is convergent for $m > 0$ and divergent for $m < 0$. By considering $m = 0$, constant thickness is obtained.

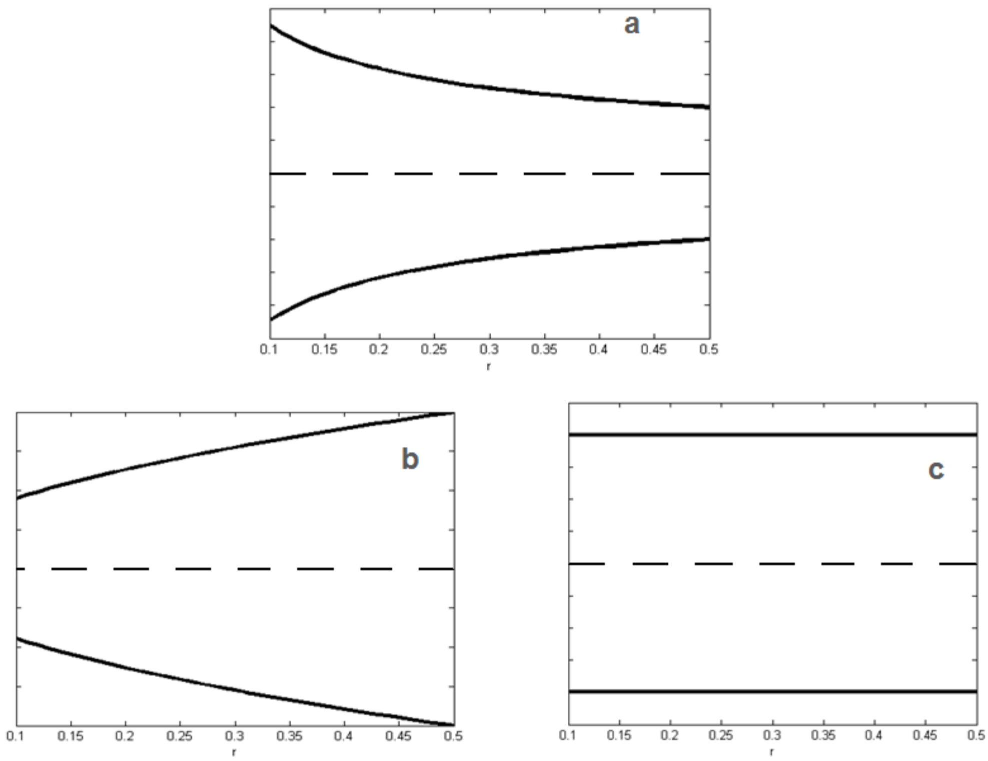

Figure 2 Thickness profile of disk (a) convergent, (b) divergent, and (c) constant.

3 Formulation of the problem

3.1 Governing equations

Despite the thickness and properties of the rotating disk, the relations between the radial displacement, u, and the strains are

$$\begin{cases} \varepsilon_r = \dfrac{du}{dr} \\ \varepsilon_\theta = \dfrac{u}{r} \end{cases} \tag{4}$$

Where ε_r and ε_θ are the total radial and hoop strain, respectively.

For disk with cylindrical coordinate system, the stress components are defined on the differential element shown in Figure (3).

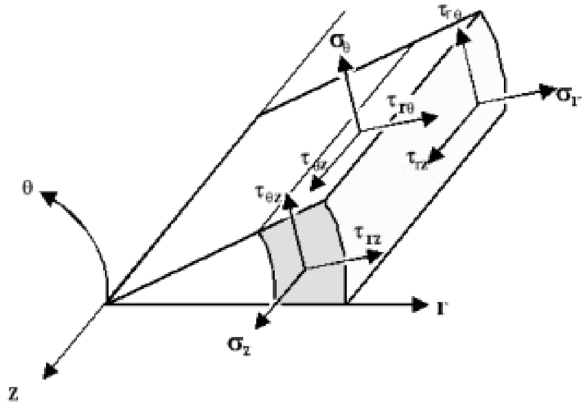

Figure3 Stress components in cylindrical coordinates.

Because the thickness of disk is considered to be small in comparison with its diameter, the problem is assumed to be plane stress. On the other hand, the inertia force due to the angular velocity of the disk is the only body force and because of symmetry, $\tau_{r\theta}$ vanishes. Thus, the equilibrium equation is reduced to [5]

The equation of equilibrium is [5]:

$$\frac{d}{dr}(hr\sigma_r) - h\sigma_\theta + \rho hr^2\omega^2 = 0 \tag{5}$$

Where $h, \sigma_r,$ and σ_θ are thickness, radial stress and hoop stress, respectively.

The deformation of the rotating disk consists of three components: elastic (ε^e), plastic (ε^p), and thermal (ε^T) strains. Total strains are the sum of these components:

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p + \varepsilon_{ij}^T \tag{6}$$

$$\varepsilon_{ij}^e = ((1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij})/E, \tag{7}$$

$$\varepsilon_{ij}^p = \phi(\sigma_{ij} - \delta_{ij}\sigma_{kk}/3), \quad \phi = 3\varepsilon_{eq}^p/2\sigma_{eq} \tag{8}$$

$$\varepsilon_{ij}^T = \alpha\Delta T \tag{9}$$

ε_{eq}^p is the equivalent plastic strain and σ_{eq} is the equivalent stress. Our analysis is based on von Misses yield criterion and σ_{eq} is given by:

$$\sigma_{eq} = \sqrt{\sigma_r^2 + \sigma_\theta^2 - \sigma_r\sigma_\theta} \tag{10}$$

Eq. (6) can be rewritten in the following form:

$$\varepsilon_{ij}^{tot} = \frac{1+\nu_{eff}}{E}\sigma_{ij} - \frac{\nu_{eff}}{E_{eff}}\sigma_{kk}\delta_{ij} + \alpha\Delta T \tag{11}$$

In this work, an elastic linear hardening [5] model is used for modeling the stress–strain relationship of the material (Figure (4)).

$$\begin{cases} \varepsilon = \dfrac{\sigma}{E} & \sigma \le \sigma_0 \\ \varepsilon = \dfrac{\sigma_0}{E} + \dfrac{1}{E_t}(\sigma - \sigma_0) & \sigma > \sigma_0 \end{cases} \tag{12}$$

Where σ_0 and E_t are the yield strength of the material and tangent modulus, respectively.

Therefore following relation is obtained for total strain:

$$\varepsilon_{ij}^{tot} = \left(\frac{1+\nu}{E} + \phi\right)\sigma_{ij} - \left(\frac{\nu}{E} + \frac{1}{3}\phi\right)\sigma_{kk}\delta_{ij} + \alpha\Delta T \tag{13}$$

Here, ν_{eff} and E_{eff} the effective Poisson ratio and effective Young’s modulus, depend on the final state of stress at a point and given by

$$E_{eff} = \frac{3E}{3+2E\phi}, \quad \nu_{eff} = \frac{3\nu + E\phi}{3+2E\phi} \tag{14}$$

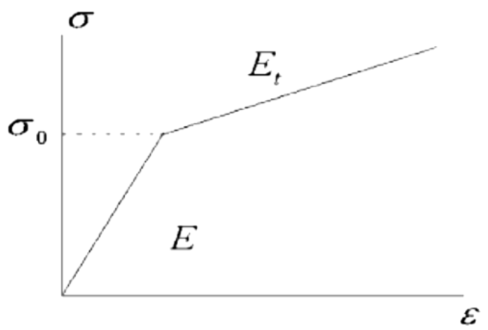

Figure 4 Idealized stress-strain curve for linear hardening materials [5]

3.2 Variable material property theory

In variable material property theory, the domain is divided into some finite uniform sub-domains in the radial direction (see Figure (5)), each annulus having the constant thermo-mechanical properties and boundary conditions with internal and external pressures as well as temperatures. Then, the form of the elastic response is used to solve elastic-plastic problems [6].

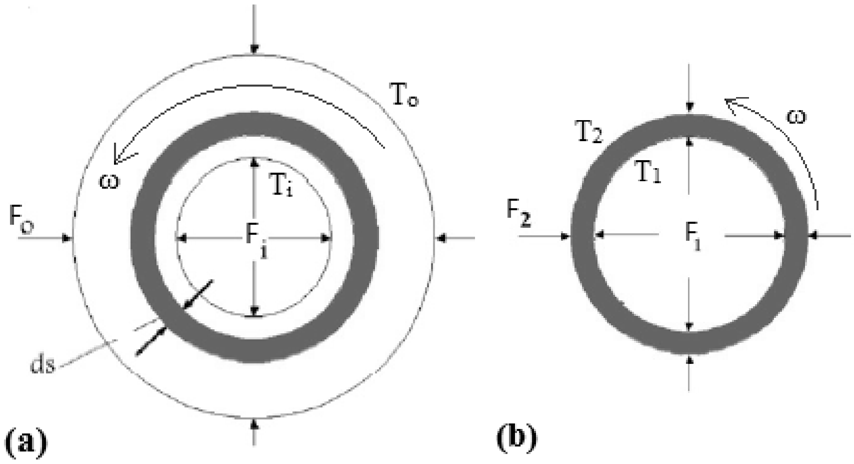

Figure 5 (a) Rotating disk with boundary condition, (b) a ring of rotating disk with its boundary condition.

In this theory, the form of the elastic solution is used to solve the inelastic problem [5].
For this purpose, first, we should obtain the thermoelastic solution of the disk.
For plane stress:

$$u(r)=\left[m_1 r+\frac{m_2}{r}\right]+\left[n_1 r+\frac{n_2}{r}-\frac{1-\nu^2}{8E}\rho r^3\omega^2\right]+\left[(1+\nu)rJ(r)+(1-\nu)Ir+r_1^2(1+\nu)\frac{I}{r}\right] \quad (15)$$

where

$$m_1=\frac{1-\nu}{Eh\left(r_1^2-r_2^2\right)}\left(F_2 r_2^2-F_1 r_1^2\right),$$

$$m_2=\frac{(1+\nu)r_1^2 r_2^2}{Eh\left(r_1^2-r_2^2\right)}\left(F_2-F_1\right)$$

$$n_1=\frac{(1-\nu)(3+\nu)}{8E}\rho\omega^2\left(r_1^2+r_2^2\right),$$

$$n_2=\frac{(1+\nu)(3+\nu)}{8E}\rho\omega^2\left(r_1^2 r_2^2\right)$$

$$I=\frac{1}{r_2^2-r_1^2}\int_{r_1}^{r_2} r\alpha T\,dr,$$

$$J(r)=\frac{1}{r^2}\int_{r_1}^{r} r\alpha T\,dr$$

where E, ν, and ρ are the mechanical and physical properties of each annulus, ω is the angular velocity of disk, h is the uniform thickness of each annulus, r_1 and r_2 are the inner and outer radiuses, F_1 and F_2 are the internal and external forces, and T is the temperature profile in each annulus.

By setting $r=r_1$ and $r=r_2$, u_1 (displacement at inner radius of each annulus) and u_2 (displacement at outer radius of each annulus) can be obtained, respectively.

Boundary conditions of disk are continuity of displacements and forces in common radiuses of neighbor strips and total boundary conditions of disk, namely:

$$\begin{cases} u(r_2^{(k)}) = u(r_1^{(k+1)}) \\ F(r_2^{(k)}) = F(r_1^{(k+1)}) \end{cases}$$
$$\begin{cases} \sigma_r = 0; \quad r = r_i \\ \sigma_r = 0; \quad r = r_o \end{cases} \tag{16}$$

where

$$\begin{cases} r_1^{(k)} = r^{(k)} - \dfrac{t^{(k)}}{2} \\ r_2^{(k)} = r^{(k)} + \dfrac{t^{(k)}}{2} \end{cases}$$

All of above matrix equations for each strip and boundary conditions in interior and exterior radiuses of each annulus and disk a set of linear equations is constituted. By solving this set of equations, displacements and forces at common boundary be obtained.

After obtaining the forces in each layer, radial and hoop stresses can be determined by:

$$\sigma_{rr} = \left[A_1 - \frac{A_2}{r^2} \right] + \frac{3+\nu}{8} \rho\omega^2 \left[r_2^2 + r_1^2 - \frac{r_1^2 r_2^2}{r^2} - r^2 \right] + E\left[-J(r) + I\left(1 - \frac{r_1^2}{r_2^2} \right) \right] \tag{17}$$

$$\sigma_{\theta\theta} = \left[A_1 + \frac{A_2}{r^2} \right] + \frac{3+\nu}{8} \rho\omega^2 \left[r_2^2 + r_1^2 + \frac{r_1^2 r_2^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2 \right] + E\left[J(r) - \alpha T(r) + I\left(1 + \frac{r_1^2}{r_2^2} \right) \right] \tag{18}$$

$$\begin{cases} A_1 = \dfrac{F_2 r_2^2 - F_1 r_1^2}{\left(r_1^2 - r_2^2\right) h}, \\ A_2 = \dfrac{\left(F_2 - F_1\right) r_1^2 r_2^2}{\left(r_1^2 - r_2^2\right) h} \end{cases}$$

Thermal loading are assumed in this form:

$$T(r) = A ln\left(\frac{r}{r_o} \right) + T_o,$$
$$A = \frac{T_i - \mathrm{T_o}}{\ln(r_i / r_o)} \tag{19}$$

To employ the VMP method, E_{eff} and ν_{eff} should be substituted in the relations of elastic solution.

To evaluate the E_{eff} and ν_{eff}, an iterative manner is needed [7]. This iterative manner will be continued until the $\sigma_{eq} - \varepsilon_{eq}$ matches on the true stress-strain curve with a small tolerance. There are three schemes to evaluate the spatial distribution of E_{eff} and ν_{eff} : (a) Projection method, (b) Arc-length method, and (c) Neuber's rule [7].

In this study, to evaluate the E_{eff} and ν_{eff} in each strip, projection method is used to update the E_{eff} and ν_{eff}. In this method, as shown in Figure (6), first, after obtaining elastic solution of disk and comparing of equivalent stress and yield stress, if σ_{eq} is greater than σ_0 (yield stress) point “a” is developed in true stress-strain curve. From point “a” with the same value of strain, point “a’” is produced by projecting on curve of plastic region of material. Then new E_{eff} is determined by $(E_{eff})_{new} = (\sigma)_{a'} / (\varepsilon)_{a'}$. By using of new E_{eff}, this procedure will be continued until the σ_{eq} - ε_{eq} matches on true stress-strain curve with a small tolerance. It should be noted that if σ_{eq} is smaller than yield stress, previous determined E_{eff} and ν_{eff} should be used.

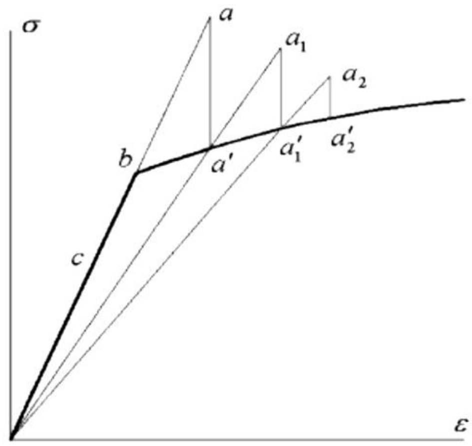

Figure 6 Projection method [4]

4 Finite element analysis

In the analysis of the disk, which thickness is small in comparison with its diameter, plane stress state is considered. In finite element analysis, commercially availabe software, ANSYS, is employed. PLANE42 elements are used to geometric model of disk. In the software, the disk is divided into some finite sub-domains in the radial direction, in which the properties are assumed to be constant. By use of APDL (ANSYS Parametric Design Language), the disk layers with different properties are modeled and boundary conditions are applied. Subsequently, the results of the finite element model are compared with the results of VMP method.

5 Numerical Results and Discussion

For numerical illustrations, one set of material mixture is considered. Theinner and outer surfaces of the FG disk are assumed to be metal-rich and ceramic-rich, respectively. Material properties of constituents in the inner and outer radiuses are presented in Table 1, referred to as mat_1 and mat_2, respectively. For this example, the disk geometric parameters are $r_i = 0.1\,\text{m}$ and $r_o = 0.5\,\text{m}$ as the inner and outer radiuses, respectively. Thickness profile is assumed to be nonlinear function of radius value of eq. (3), with $h_o = 0.02$ and $m = 0.5$.

Table 1 Properties of constituents of FG rotating disk.

	Young's modulus (GPa)	Tangent modulus (GPa)	Poisson's ratio	Density (kg/m^3)	Yield strength (MPa)	Thermal expansion coefficient ($1/°\text{C}$)
mat_1	69	27	0.34	2715	150	23×10^{-6}
mat_2	115	57	0.293	4515	1030	8×10^{-6}

We first study the convergence of the VMP theory and FE analysis for the given properties between the inner and the outer surface of the disk. It's find that for n=30 (number of divisions), in VMP theory and FE analysis, the responses are converged.

Then, in Figures (7) and (8), we obtained the effective modulus and Poisson's ratio of disk at T=350°C. In these figures, we compared the elasticity and effective modulus and Poisson's ratio. As mentioned before, if the radius of disk does not inter to plastic region, the effective and elasticity modulus will be the same. As shown in this figures, from inner radius to $r = 0.147\ m$, the disk is under plastic strain. It's obvious that in this region, duo to plasticity, effective modulus is smaller than elasticity modulus and effective Poisson's ratio is more than Poisson's ratio.

Then, by use of these values for effective modulus and effective Poisson's ratio, the results obtained from VMP theory and FE analysis are presented and compared in Figure (9) for radial and hoop stresses, respectively. As observed in these figures, these two methods are in good agreement. Maximum differences between them are 4.5% and 9% for radial and hoop stresses, respectively.

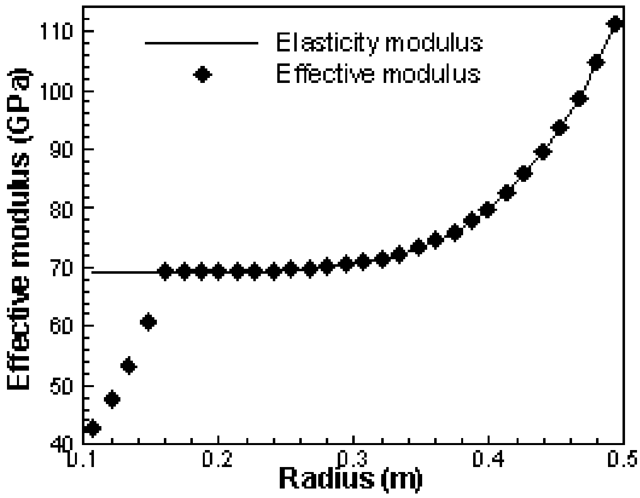

Figure 7 Effective modulus of disk at T=350°C

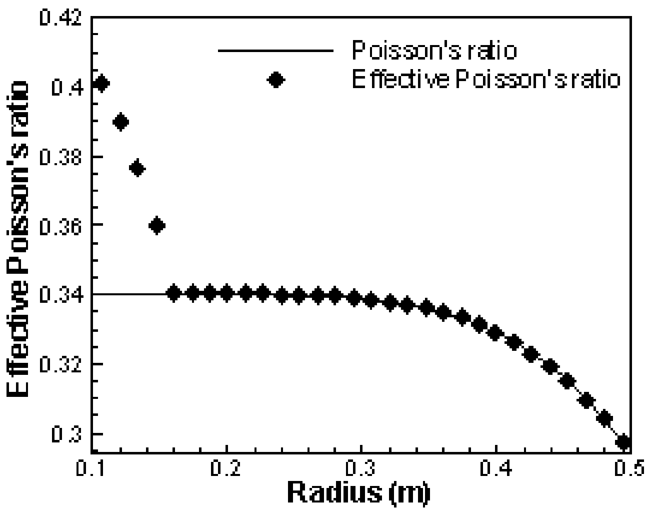

Figure 8 Effective Poisson's ratio of disk at T=350°C

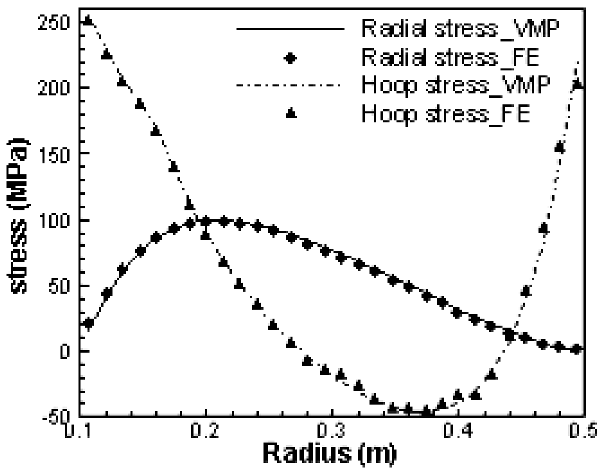

Figure 9 Comparison of VMP and FE results for stresses

In addition, radial and hoop strains of the mentioned disk are plotted in Figure (10). Clearly, the comparison is reasonably well and the maximum difference between two methods are 1.8% and 1.4% for radial and hoop strains, respectively.

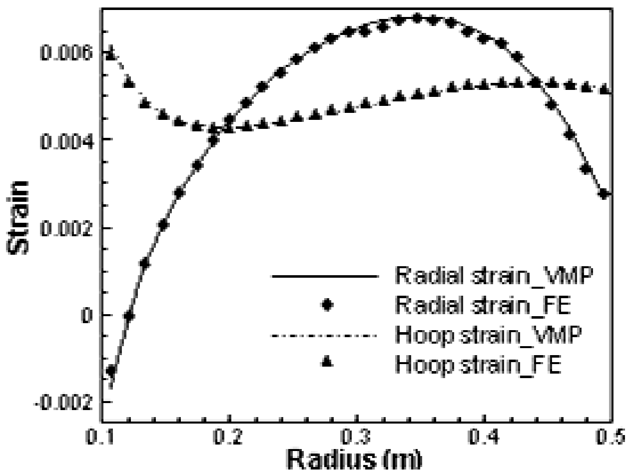

Figure 10 Comparison of VMP and FE results for strains

Radial displacement of the disk are plotted in Figure (11). Clearly, the comparison is reasonably well and the maximum difference between two methods is 0.16%.

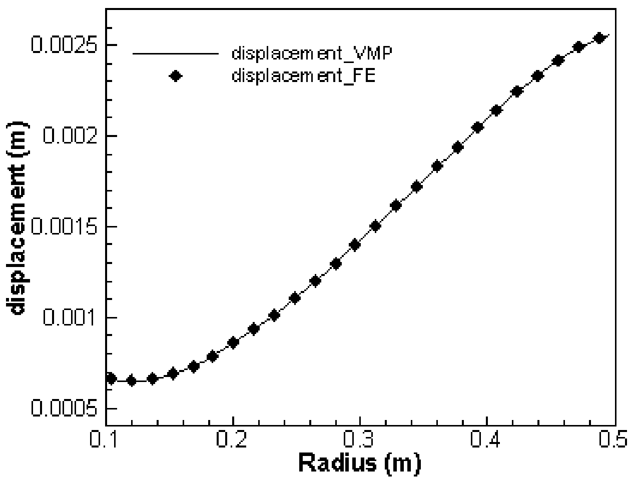

Figure11 Comparison of displacement between the finite element method and VMP

The effect of temperature gradient on the effective Young's modulus, radial stress, and hoop stress has been shown in Figures (12-14), respectively. For simplicity of numerical calculations, the temperature at the inner surface remains unchanged, i.e. $T_i = 0°C$; and the temperature at the outer surface is changed.

The previous research on the uniform rotating disk reveals that the yielding initiates from the inner radius [5], while as is presented in Figure (12), plasticity in FG disk could be initiated at any point.

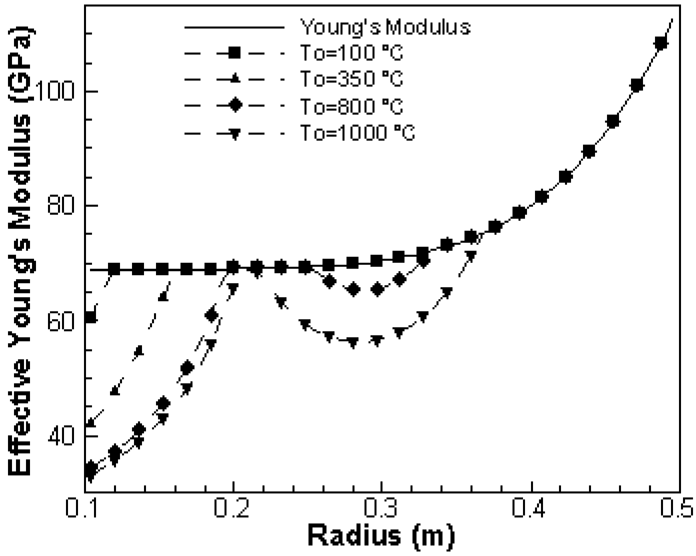

Figure 12 Effect of temperature gradient changes on the effective Young's modulus of FG rotating disk.

From Figure (13), it can be seen that in lower temperature gradient, the value of hoop stress at the inner radius are larger than that at the outer radius and with increases in temperature gradient, the maximum value of hoop stress occurs at the outer surface of the disk.

Furthermore, there exists two points on the domain, i.e., $r = 0.216\,\text{m}$ and $r = 0.46\,\text{m}$ in which the changes in the temperature gradient have no effect on the hoop stresses.

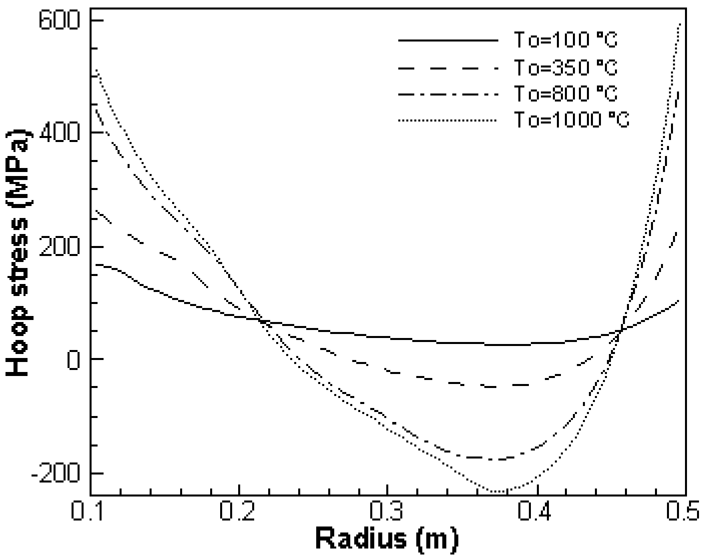

Figure 13 Effect of temperature gradient changes on the hoop stresses of FG rotating disk.

From Figure (14), it is found that the radial stresses increase as the temperature gradient between the inner and outer surfaces of the disk increases; and the location of maximum stress approaches to the inner radius. It can also be seen that radial stresses have the same value at $r = 0.4\,\mathrm{m}$. That means there is an interior radius in the disk in which the temperature gradient has the least effect on radial stresses. Passing through this point, the effect of temperature gradient on radial stress reverses.

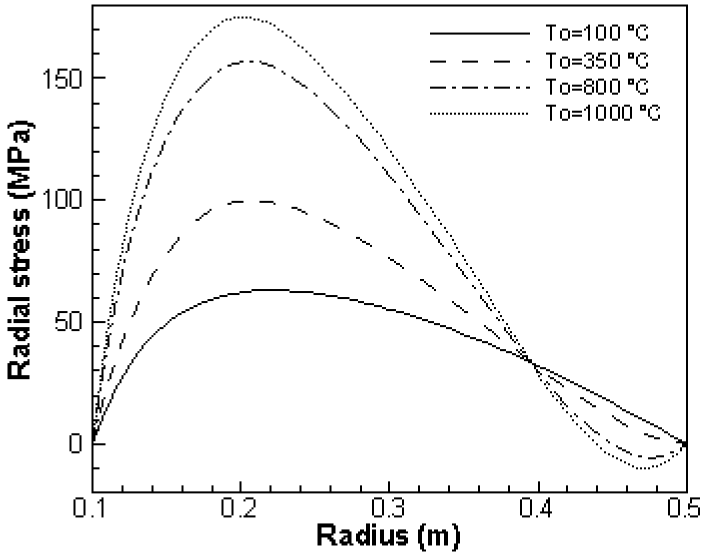

Figure 14 Effect of temperature gradient changes on the radial stresses of FG rotating disk.

Figure (15) shows the effect of different profile thickness on radial stress with changing m in eq. (3). In this hyperbolic profile, for $m < 0$, the profile is divergent and for $m > 0$, the profile is convergent. The constant thickness is obtained for $m = 0$. This figure shows that the disk with divergent or convergent profile, have smaller maximum radial stress than that with constant thickness. It can also be seen that for both divergent and convergent thickness profile of disk, the radial stress decreases as the order of hyperbola, m, increases.

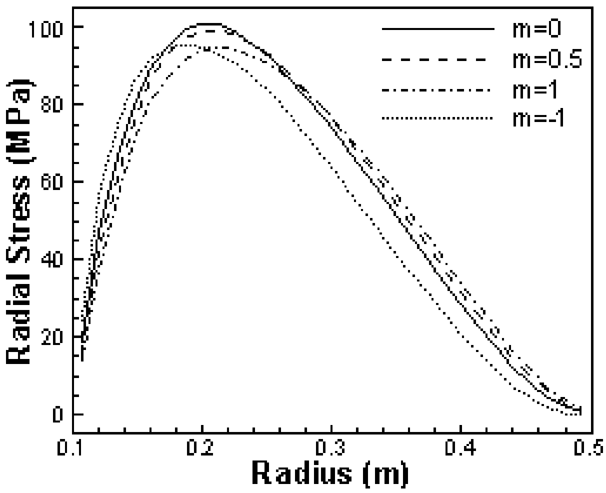

Figure 15 Effect of disk profile on radial stress

6 Conclusions

In this paper, thermoplastic analysis of FGM rotating disk with variable thickness is solved by using the variable material property theory. In this method, we use of elastic response to solve the inelastic problem, with substitution appropriate effective modulus that obtained from true stress-strain curve. The results obtained by the VMP method are then compared with the results obtained by the finite element analysis using ANSYS software. The results reveal that mentioned methods are in very good agreement in both elastic and plastic states.

Finally, the effect of temperature gradient and thickness profile on the stress behavior of disk was investigated. The results show that these parameters have significant effects on stress behavior of the disk and discs with variable thickness profile have smaller stresses than those with constant thickness.

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Nomenclature

E	Young's modulus
E_t	tangent modulus
F	force per unit of length
h	thickness profile
h_o	thickness at the axis of disk
m	geometry property
mat_1	refer to material at the inner radius of disk
mat_2	refer to material at the outer radius of disk
N	grading index
r	refer to radial direction
r_i	inner radius of disk
r_o	outer radius of disk
P_c	material property of ceramic
P_f	FG effective property
P_m	material property of metal
u	radial displacement
V_c	ceramic volume fraction
V_m	metal volume fraction

Greek symbols

ΔT	thermal gradient
ε^e	elastic strain
ε^p	plastic strain
ε^T	thermal strain
ε_r	radial strain
ε_θ	hoop strain
θ	refers to hoop direction
ν	Poisson's ratio
ρ	mass density
σ_0	yield strength
σ_r	radial stress
σ_θ	hoop stress
ω	angular velocity of disk

چکیده

در این مقاله، تحلیل ترموپلاستیک دیسک های دوار FGM با ضخامت متغیر و تحت سرعت زاویه ای یکنواخت ارائه می شود. پاسخ ها با استفاده از تئوری خواص مادی متغیر (VMP) بدست می آید. در این تئوری، ابتدا دیسک مورد بررسی به چندین زیر محدوده در راستای شعاعی تقسیم بندی می شود. در این زیر محدوده ها خواص ثابت فرض می شود و از فرم پاسخ الاستیک برای برای حل مسائل الاستیک-پلاستیک استفاده می شود. این نتایج با نتایج به دست آمده از تحلیل اجزای محدود حاصل از نرم افزار انسیس مقایسه می شود. این نتایج نشان می دهد که روش های مذکور هم در حالت ترموالاستیک و هم در حالت ترموپلاستیک، تطابق خوبی با یکدیگر دارند. در پایان، تاثیر پارامترهای متعدد شامل توزیع حرارتی و پروفیل ضخامت بر رفتار تنشی دیسک بررسی می شود.