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Vibrational Analysis of Cables using the Non Standard Finite Difference Method

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In continuous systems such as rods, cables, and shafts, the differential equation governing the behaviour of the structure can be obtained by applying Newton's second law. Only a limited group of these equations have an analytic solution; hence the provision of efficient and stable numerical methods is of particular importance. The method used in this study is a nonstandard finite difference (NSFD) scheme, which is an effective numerical method for solving the above-mentioned equations. One of the important features of this method is its relatively high accuracy. Results obtained by nonstandard finite difference methods are more compatible with the exact behaviour of the problem than that of the standard finite difference method. In this research, the equations governing the vibrations of cables with three different boundary conditions are solved. Three solutions including the analytical solution, the standard finite difference method (SFD) and the NSFD method are compared. Results show that the error of the NSFD is significantly less than of the SFD.

Keywords: vibrations, nonstandard finite difference, cable, numerical methods

1 Introduction

The differential equations are usually involved in many branches of physics and engineering. The numerical solution of the algebraic differential equations has been greatly studied during recent years and much research have been carried out to develop such solutions. Many analytical and numerical methods such as Hirota’s two path scheme [1], symmetrical equations [2], homotopy perturbation [3], variable repetition [4], reversible derivative [5], Runge-Kutta [6], Pad estimations [7], and other numerical methods [8,9] are used for engineering problems. Qureshi et al. [10] used the homothopy scheme to solve the multi-point boundary value problems. They established an approximate method in a way that satisfies the boundary conditions exactly. Sumasolik and Wedagedera [11] used the finite difference method to solve the mathematical model of the corals’ growth inside a reservoir.
They assumed the numerical model to be one dimensional and without heat flux boundary condition. The obtained numerical results were conformal to the analytic solution. Majeed et al. [12] studied the boundary layer transfer flow and the heat transfer in a fluid within the ferromagnetic dust particles on a tensioned flat plate regarding the magnetic bipolar effects.

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By simplifying the non-linear equations to linear equations, they solved the governing differential equation using the Runge Kutta scheme. In recent years, some partial differential equations have been analyzed and solved using the Adomin method [13]. Shakeri and Dehghan [14] used Adomin method in order to solve a system of equations including the differential and integral equations, which are of fundamental importance in biology. They solved the equations that describe the biologic model of the living species. Biazar et al. [15], introduced a new type of Adomin scheme. They obtained a general continuous method that does not require the Adomin polynomials' calculation. They also compared the new method with both conventional method of Adomin's separation and the variable repetition. Recently, this method has been used for solving the vibration problems in the structural and mechanical systems [16]. One of the powerful numerical methods in solving the differential equations is the finite difference method. Despite its accuracy, this method poses a significant challenge of choosing the step size. If the step size is selected to be small, the computational cost is significantly increased. On the other hand, if the step size is taken large, the results become less accurate and sometimes leads to divergent results. In order to solve this drawback, Mickens [17] introduced the nonstandard finite difference (NSFD) method. The basic idea of the nonstandard method stems from accurate difference methods. Thanks to this method, one could choose the length of the step size in a more efficient and suitable way compared to the standard finite difference method. The golden step size is indeed a function (denominator function) that gets the assumed step size as the input data and converts it to the efficient step size. At first, Mickens and Washington [18], and Rager and Mickens [19] used the above method to solve some differential equations. Afterward, González et al. [20] used the NSFD method for the numerical solution of the biologic and population models. Moaddy et al. [21] used the nonstandard finite difference method for the fractional partial differential equations. Also, Ongun and Turhan [22] compared the nonstandard and standard numerical methods for the ordinary differential equation system governing the replicating process of AIDS virus. Cresson and Pierret [23] studied the NSFD scheme for the general state of two dimensional differential equations including various models in a dynamic population based on the non-local estimations. Regarding the discretization of the parameters, they proved the convergence of their method with retaining the constant points, and stability of the system. Zhang et al. [24] presented their algorithm of finite difference method for solving the coupled Burger's viscous problems, which used the accurate solution within the NSFD based on the standard finite difference method. One of the methods which leads to the suitable denominator function in nonstandard method is the accurate discretized method which is used in this study. Mickens et al. [25] assumed a second order linear equation and presented an accurate discretization method for it. Roeger [26] solved a linear system using the NSFD method. Roeger [27] also studied a two dimensional linear system and presented an accurate finite difference method for the system. In this research, the vibrational equations of the continuous systems such as cables are analyzed through analytic and approximate methods. The solution of the differential equation presents the displacement of the various points on the cable, depending on two variables of location and time. Also in this research, a new computational viewpoint, which is described as a NSFD is used to solve the wave equation. However, this method is categorized as a general SFD condition. Despite having the exact solution, we still emphasize on solving the equations using the numerical scheme. This argument is addressed in the following discussion. Generally, the new numerical schemes should be examined across some factors to check their validity and performance. It is obvious that in this process, one cannot compare them with the other numerical scheme, because every numerical scheme has its own error due to its nature. Therefore, numerical solutions should be compared with the exact numerical solutions.

In this paper, a cable with three boundary conditions and with sinusoidal initial condition is considered and is solved using three methods of analytical, SFD and NSFD. These examples show the effectiveness of the NSFD method compared to the SFD method because the error of NSFD is less than of SFD.

2 Theoretical fundamentals of the problem

The analytical solution for the cable with the sinusoidal initial condition and 3 various BCs: The governing equation of the cable is solved using the analytical method:

2-1 Analytical method
2-1-1 Free Vibrations of a Fixed-Fixed end Cable

A stretched cable of length l is considered as shown in figure (1). For this uniform cable with a constant tension of P, the free vibrational equation is as follows:

$$P\frac{\partial^2 u(x,t)}{\partial x^2} = \rho\frac{\partial^2 u(x,t)}{\partial t^2} \tag{1}$$

or

$$c^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \tag{2}$$

in which:

$$c = (\frac{P}{\rho})^{\frac{1}{2}} \tag{3}$$

and ρ is the density of the cable, u is the vertical displacement, x is the horizental axis and t is the time. The BC and initial conditions are:

$$u(0,t) = u(l,t) = 0 \tag{4}$$

$$\frac{\partial u(x,0)}{\partial t} = 0 \quad , \quad u(x,0) = \sin \pi x \tag{5}$$

If $c^2 = 1$, the exact solution of the above equation is:

$$u(x,t) = \sin \pi x \cos \pi t \tag{6}$$

2-1-2 Free Vibrations of a one free end Cable

For this problem the BC is as follows:

$$\frac{\partial u(0,t)}{\partial x} = u(l,t) = 0 \tag{7}$$

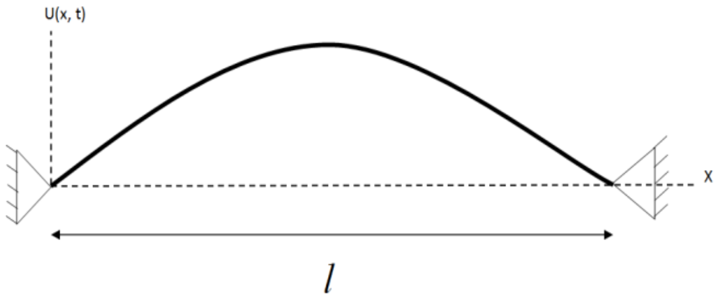

Figure 1 The fixed-fixed end cable with sinusoidal initial condition

The other relations are such as previous case. For the exact solution we have:

$$u(x,t)=\sum_{n=0}^{\infty} C_n \cos\omega_n x \cos\omega_n t$$
$$\omega_n=\frac{(2n+1)\pi}{2l} \qquad n=0,1,2,3,..$$
$$C_n=\frac{1}{l}\left\{\begin{array}{l}\frac{1}{\pi-\omega}\left[1-\cos(\pi-\omega)l\right] \\ +\frac{1}{\pi+\omega}\left[1-\cos(\pi+\omega)l\right]\end{array}\right\} \tag{8}$$

2-1-3 Free Vibrations of a Free-Free end Cable

The BC is as follows:

$$\frac{\partial u(0,t)}{\partial x}=\frac{\partial u(l,t)}{\partial x}=0 \tag{9}$$

Regarding equation (2), the exact solution is similar to the solution of the Fixed-Free end cable except the natural frequency which is:

$$\omega_n=\frac{n\pi}{l} \qquad n=0,1,2,3,.. \tag{10}$$

2-1 The SFD method for the problem

In this section, the governing equations of the cable's free vibrations is solved using the SFD method.

2-2-1 Fixed-Fixed Cable

Using the central difference, the second derivative of time is:

$$\frac{\partial^2 u(x,t)}{\partial t^2}=\frac{u_i^{n+1}-2u_i^n+u_i^{n-1}}{(\Delta t)^2} \tag{11}$$

Where *i* and *n* are the points located in a 2-D plane and in a time instances. Furthermore, the average of the central approximation is used for the second derivative of the coordinates in of *n-1* and *n+1* grid surface*:*

$$\frac{\partial^2 u(x,t)}{\partial x^2})_i^n = \frac{1}{2}(\frac{\partial^2 u(x,t)}{\partial x^2})_i^{n-1} + \frac{1}{2}(\frac{\partial^2 u(x,t)}{\partial x^2})_i^{n+1} = \frac{1}{2(\Delta h)^2}(u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1} + u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) \quad (12)$$

According to the main equation of the free vibrations of the cable, the equality of equation (11) and (12) leads to:

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2} = \frac{1}{2(\Delta h)^2}(u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1} + u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) \quad (13)$$

For the simplification, it is assumed that:

$$\frac{(\Delta t)^2}{(\Delta h)^2} = r$$

Eventually, rearranging equation (13) we have:

$$(2+2r)u_i^{n+1} - r(u_{i-1}^{n+1} + u_{i+1}^{n+1}) = r(u_{i-1}^{n-1} + u_{i+1}^{n-1}) - (2+2r)u_i^{n-1} + 4u_i^n \quad (14)$$

Using the initial condition:

$$\frac{\partial u(x,0)}{\partial t} = \frac{u_i^1 - u_i^{-1}}{2\Delta t} = 0 \Rightarrow u_i^1 = u_i^{-1} \quad (15)$$

With applying the discretization method for central time-location [1], we have:

$$u_i^{n+1} = 2(1-r)u_i^n + r(u_{i+1}^n + u_{i-1}^n) - u_i^{n-1} \quad (16)$$

If in the above equation *n=0* ($t = 1\Delta t$), then the following equation is derived:

$$n = 0 \Rightarrow u_i^1 = 2(1-r)u_i^0 + r(u_{i+1}^0 + u_{i-1}^0) - u_i^{-1} \quad (17)$$

With the substitution of equation (15) in (17), we have:

$$u_i^1 = (1-r)u_i^0 + \frac{r}{2}(u_{i+1}^0 + u_{i-1}^0) \quad (18)$$

By using of equation (18), one can calculate the whole points located on the surface *n=1*. As an example, with assumption of $l=1$ and dividing the distance into 4 equal parts (local step size of 0.25), we have:

$$\begin{aligned} i=1 &\Rightarrow u_1^1=(1-r)u_1^0+\frac{r}{2}(u_2^0+u_0^0) \\ i=2 &\Rightarrow u_2^1=(1-r)u_2^0+\frac{r}{2}(u_3^0+u_1^0) \\ i=3 &\Rightarrow u_3^1=(1-r)u_3^0+\frac{r}{2}(u_4^0+u_2^0) \end{aligned} \tag{19}$$

Regarding the BC, we have $u_0^1=u_4^1$ and $u_0^n=u_4^n$. Now, the location of the other points in the other time step $t=\Delta t, t=2\Delta t$ etc can be calculated. For example, with choosing *n=0* in equation (14), we have:

$$(2+2r)u_i^2-r(u_{i-1}^2+u_{i+1}^2)=r(u_{i-1}^0+u_{i+1}^0)-(2+2r)u_i^0+4u_i^1 \tag{20}$$

With the substitution of various *i* values in the above equation, we have:

$$\begin{aligned} i=1 &\Rightarrow (2+2r)u_1^2-ru_2^2=ru_0^2+ru_0^0+ru_2^0-(2+2r)u_1^0+4u_1^1 \\ i=2 &\Rightarrow (2+2r)u_2^2-ru_1^2-ru_3^2=ru_1^0+ru_3^0-(2+2r)u_2^0+4u_2^1 \\ i=3 &\Rightarrow (2+2r)u_3^2-ru_2^2=ru_2^0+ru_4^0-(2+2r)u_3^0+4u_3^1+ru_4^2 \end{aligned} \tag{21}$$

The left-hand side of the equations show the unknown variables and their right-hand side show the known values. It can be observed that in this especial example, which divides the interval between 0 to 1 into 4 parts, in (21) three equations of three unknowns are created. The solution of these equations help us to calculate the whole points of the surface *n=2*.
The higher surfaces of *n* could be calculated accordingly.

2-2-2 Fixed-Free Cable

For this condition, equations (11) to (14) will be still applicable. In the following, regarding the BC in $x=0$, we have:

$$\frac{\partial u(0,t)}{\partial x}=0 \Rightarrow \frac{u_1^n-u_{-1}^n}{2h}=0 \Rightarrow u_1^n=u_{-1}^n \tag{22}$$

If the interval of 0 to $l=1m$ is divided into 4 parts (similar to previous case), we have 5 points in the horizontal axis. Using equations (16) and (22) results in:

$$\begin{aligned} & i=0 \Rightarrow u_0^1=(1-r)u_0^0+\frac{r}{2}(u_1^0+u_{-1}^0) \\ & \Rightarrow u_0^1=(1-r)u_0^0+\frac{r}{2}(2u_1^0) \qquad i=1 \Rightarrow u_1^1=(1-r)u_1^0+\frac{r}{2}(u_2^0+u_0^0) \\ & i=2 \Rightarrow u_2^1=(1-r)u_2^0+\frac{r}{2}(u_3^0+u_1^0) \quad i=3 \Rightarrow u_3^1=(1-r)u_3^0+\frac{r}{2}(u_4^0+u_2^0) \end{aligned} \tag{23}$$

According to BC of $u(l,t)=0$, we have $u_4^1=0$. The solution continues similar to the previous case.

2-2-3 Free-Free Cable

Similar to the previous case, equations (11) to (14) and (22) will be applicable. From BCs, we have:

$$\frac{\partial u(l,t)}{\partial x}=0\Rightarrow\frac{u_{m+1}^{n}-u_{m-1}^{n}}{2h}=0 \\ \Rightarrow u_{m+1}^{n}=u_{m-1}^{n} \tag{24}$$

In this equation, to represent a general condition, the interval is divided into *m* element.
As an example, if the location interval is divided into 4 parts (*m*=4), it has 5 points. Applying equations (16), (22) and (24), the vertical displacement of these points can be calculated as:

$$\begin{aligned} & i=0\Rightarrow u_0^1=(1-r)u_0^0+\frac{r}{2}(u_1^0+u_{-1}^0) \\ & \Rightarrow u_0^1=(1-r)u_0^0+\frac{r}{2}(2u_1^0) \\ & i=1\Rightarrow u_1^1=(1-r)u_1^0+\frac{r}{2}(u_2^0+u_0^0) \\ & i=2\Rightarrow u_2^1=(1-r)u_2^0+\frac{r}{2}(u_3^0+u_1^0) \\ & i=3\Rightarrow u_3^1=(1-r)u_3^0+\frac{r}{2}(u_4^0+u_2^0) \\ & i=4\Rightarrow u_4^1=(1-r)u_4^0+\frac{r}{2}(u_5^0+u_3^0) \\ & \Rightarrow u_4^1=(1-r)u_4^0+\frac{r}{2}(2u_3^0) \end{aligned} \tag{25}$$

The rest of the solution is similar to the first case.

2-2 The solution of the problem using the NSFD method

The issue of NSFD method is assessed using the discretization of the first-order differential equation as:

$$\frac{dx}{dt}=\frac{x_{k+1}-x_k}{h} \tag{26}$$

where *h* is the step size. As it was mentioned in the introduction, in the standard method, the length is chosen randomly. By defining the denominator function, Mikens [17] presented a new scheme to select the step size:

$$\frac{dx}{dt}=\frac{x_{k+1}-x_k}{\varphi(h)} \tag{27}$$

The denominator function $\varphi(h)$ operates in a way that it gets the initial pitch length from the user and then transforms it to a better or the golden pitch length. The denominator function should satisfy the following conditions:
a) The order of accuracy should be the same as the accuracy of the equation.
b) It should be a real and positive function.
c) The denominator function should be an ascending function.

In general case, it is observed that the existing parameters in the main equation also appear in the denominator function. Some of the functions, which satisfy the above conditions, include the following functions:

$$\varphi(\mathrm{h}) = \mathrm{h}, \quad \sinh, \quad \mathrm{e}^{\mathrm{h}} - 1, \quad \frac{1 - e^{-\lambda h}}{\lambda} \tag{28}$$

λ is a parameter in the equation that should be solved.
For determining the denominator function, there are various methods. One of these methods is using the analytical solution of the equation to compute the denominator function. Jang et al. [24] used this method to obtain the denominator function. In the current study, this method is used to find the denominator function.
With consideration of $\frac{(\Delta t)^2}{(\Delta h)^2} = r$, two denominator functions should be selected for the step size and time step in a way that they cannot eliminate each other.
Assuming $c^2 = 1$, the governing equation will be $U_{xx} = U_{tt}$. In this step, the equation is re-derived using the forward and backward derivatives. So we have:

$$\partial_x u(x,t) = \frac{u(x+h,t) - u(x,t)}{\psi_1} \tag{29}$$

$$\bar{\partial}_x u(x,t) = \frac{u(x,t) - u(x-h,t)}{\psi_2} \tag{30}$$

$$\partial_t u(x,t) = \frac{u(x,t+\Delta t) - u(x,t)}{\varphi_1} \tag{31}$$

$$\bar{\partial}_t u(x,t) = \frac{u(x,t) - u(x,t-\Delta t)}{\varphi_2} \tag{32}$$

Now using the relations:

$U_{xx} = \partial_x \bar{\partial}_x u(x,t)$ and $U_{tt} = \partial_t \bar{\partial}_t u(x,t)$

results in:

$$U_{xx} = \partial_x \bar{\partial}_x U = \left(\frac{u(x+h,t) - u(x,t)}{\psi_1}\right)\left(\frac{u(x,t) - u(x-h,t)}{\psi_2}\right) \tag{33}$$

$$U_{tt} = \partial_t \bar{\partial}_t U = \left(\frac{u(x,t+\Delta t) - u(x,t)}{\varphi_1}\right)\left(\frac{u(x,t) - u(x,t-\Delta t)}{\varphi_2}\right) \tag{34}$$

In cases such as [24], while calculating the above differences, some functions are created, which are indeed non-standard method denominator functions. In such conditions, since the final answer is exponential, both sides of the equation can be simplified. Therefore, finding the denominator function is easier. Whereas in the current study, the final answer cannot be presented in exponential form. Therefore, the higher differences are formed and the idea of obtaining the denominator function is extracted from the final answer:

$$\frac{1}{U(x,t)}=\frac{1}{\sin\pi x\cos\pi t} \tag{35}$$

$$\frac{1}{U(x+h,t)}=\frac{1}{\sin\pi(x+h)\cos\pi t} \tag{36}$$

$$\frac{1}{U(x,t)}-\frac{1}{U(x+h,t)}=\frac{1}{U(x+h,t)}(2\cos\pi h-1) \tag{37}$$

$$\frac{1}{U(x,t)}-\frac{1}{U(x-h,t)}=\frac{1}{U(x-h,t)}(-1) \tag{38}$$

$$\frac{1}{U(x,t)}-\frac{1}{U(x,t+\Delta t)}=\frac{1}{U(x,t+\Delta t)}(1-\cos\pi t+\tan\pi t\sin\pi t) \tag{39}$$

$$\frac{1}{U(x,t)}-\frac{1}{U(x,t-\Delta t)}=\frac{1}{U(x,t-\Delta t)}(\cos\pi t+\tan\pi t\sin\pi t-1) \tag{40}$$

The reason of these calculations is to achieve the appearance of the denominator function. Therefore, this method is followed close to the *h* step size.

Every coefficient functions of right-hand side of the equations (37) to (40), can be selected as the denominator function. The important point is that the order of accuracy of these functions should be of $h^2+0(h^4)$ order. Therefore, with the addition of the appropriate coefficients, this order of accuracy should be attained. After a few trial and error, the appropriate denominator functions are introduced as:

$$\phi(t)=\frac{(1-\cos\pi t+\tan\pi t\sin\pi t)}{1.5\pi^2} \tag{41}$$

$$\phi(h)=\frac{(2-2\cos\pi h)}{\pi^2} \tag{42}$$

which are substituted for Δt^2 and Δh^2, respectively.

3 Results

3-1 Fixed-Fixed Cable

figure (2) shows the accurate solution of the displacements for $l=1m$:

Regarding the BCs in this Figure, it is observed that the cable has no displacement in both fixed ends. With considering $\Delta h=0.02$ and $\Delta t=0.01$, and the final time of $T=1s$, the standard and nonstandard solutions are shown in Figures (3 & 4), respectively:

In these Figures, the standard and non-standard answers are generally similar to the exact solution confirming the validation of the approximate methods. The error is defined as the difference between each result and the exact solution.

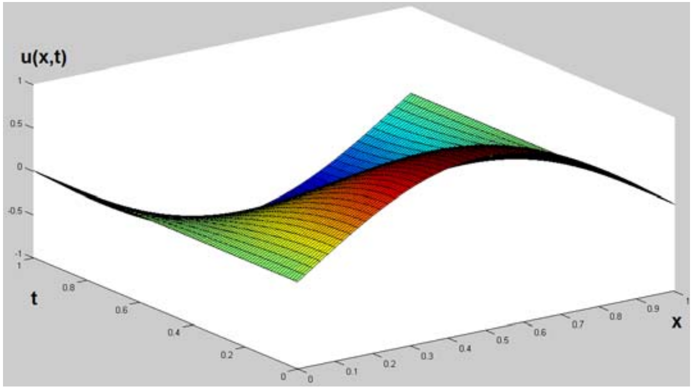

Figure 2 The displacement of various points of Fixed-Fixed cable using the analytical method

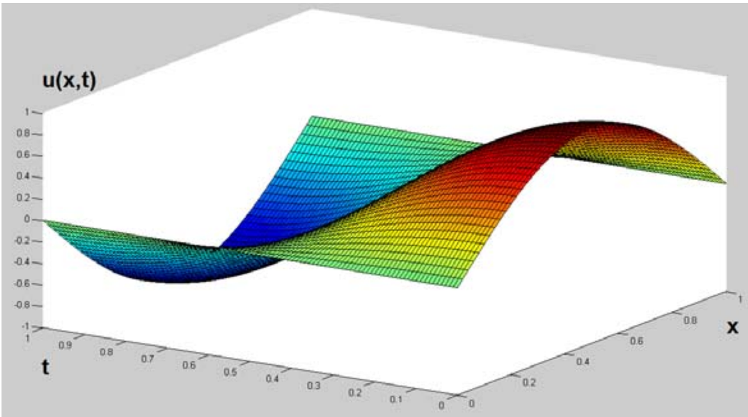

Figure 3 The displacement of various points of Fixed-Fixed cable using the standard methods

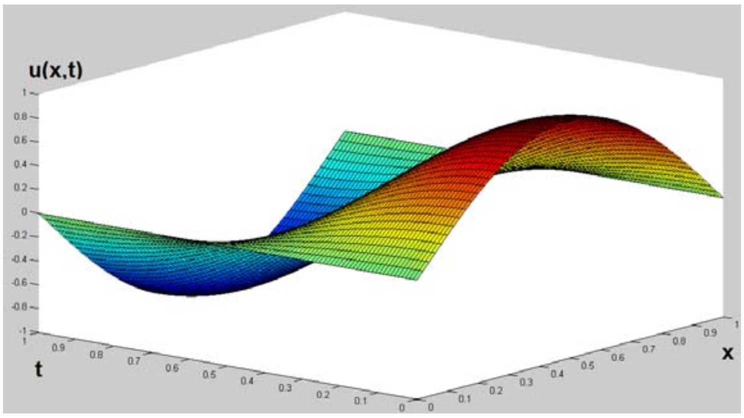

Figure 4 The displacement of various points of Fixed-Fixed cable using the non-standard methods

Figures (5&6) show the error of standard and nonstandard methods.
It can be seen from Figures (5&6) that the maximum error in the standard method is 7×10^{-4} and in non-standard method is 2.5×10^{-4}. In the general case, there are 3 matrices of exact, standard and the nonstandard solutions, which are shown by $[U_{exact}]$, $[U_{SFD}]$ and $[U_{NSFD}]$, respectively. Every element of the above matrices shows a value of vertical displacement $u(x,t)$. Therefore, subtracting $[U_{SFD}]$ from $[U_{exact}]$, means elements of the vertical displacements are subtracted and the error of the standard method is determined with respect to the exact solution. So we have:

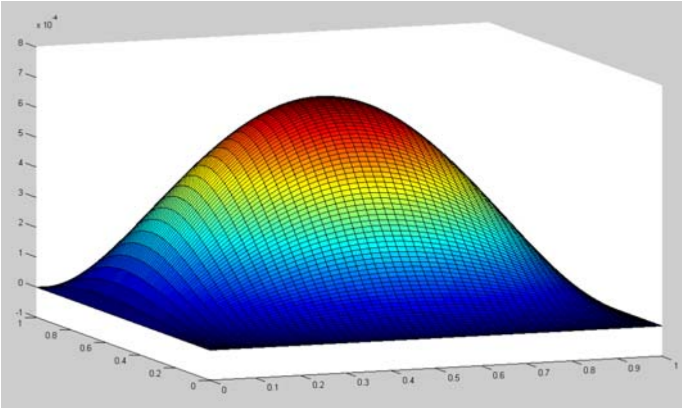

Figure 5 The error of the standard methods for the fixed-fixed cable

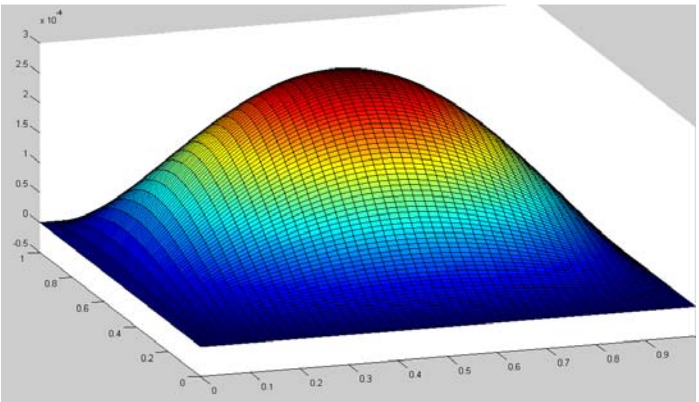

Figure 6 The error of the nonstandard for the fixed-fixed cable

$$[Error_{SFD}] = [U_{exact}] - [U_{SFD}] \tag{43}$$

$$[Error_{NSFD}] = [U_{exact}] - [U_{NSFD}] \tag{44}$$

To compare the error of two standard and non-standard methods, the sum of squared errors (SSE) is used. Therefore, the SSE parameter is defined, which is the sum of square elements of the error matrix. The less the SSE, the less the error value is and showing the solution method is more desirable.
The value of the SSE in fig (5) is 4.6633×10^{-4} and in fig (6) is 8.9375×10^{-5}, which shows %81 decrease in nonstandard method's error compared to the standard method (relative error). The decrese is calculated as eq. (45):

$$\frac{\mathrm{SSE_{SFD}} - SSE_{NSDF}}{\mathrm{SSE_{SFD}}} \times \%100 \tag{45}$$

To show the advantage of the nonstandard method, some especial points can be considered inside the solution zone and the error value of the various methods are compared [28].
If the above approach is set as the basis of the analysis, it can be shown that the amount of error decrease is %350 in this example, because the maximum standard method's error is also decreased from 7×10^{-4} to 2.5×10^{-4} in non-standard method.

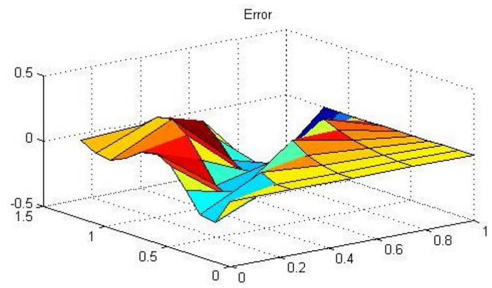

Figure 7 The error of standard method for the fixed-free cable

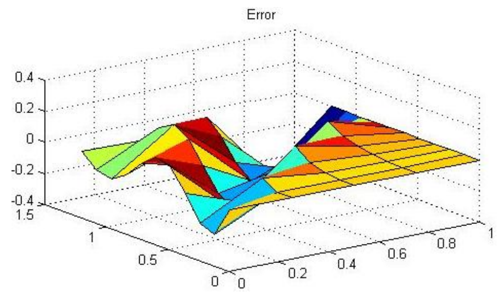

Figure 8 The error of nonstandard methods for the fixed-free cable

3-2 Fixed-Free end cable

For this problem, it is assumed that $\Delta h = 0.25$, $\Delta t = 0.12$ and $T = 1.2s$. The error of standard and nonstandard methods are ploted in Figures (7&8).
For this example, according to Figures (7&8), the absolute value of the maximum error for standard and nonstandard methods are 0.41 and 0.36, respectively showing a lower error value of the nonstandard method. Comparing the sum square error for these methods again approves the efficiency of the nonstandard method. The SSE values are:
SSE_{SFD}=1.0227 , SSE_{NSFD} =0.7718

3-3 Free-Free end cable

For the free-free cable, the step size, time step and the final time are considered as:
$\Delta h = 0.25$ $\Delta t = 0.12$ ، , $T = 1.2s$

The errors are shown in Figures (9&10).
For Figures (9&10), the calculated SSE values are:
SSE_{SFD}=1.8453, SSE_{NSFD} =1.5538
Regarding the sum squared error, it can be shown that the error of NSFD is less than the SFD one. To show the advantages of the nonstandard method, the values of the sum square error for both methods of standard and nonstandard for three BCs of a cable are shown in table (1).
According to table, it is shown that for the fixed-fixed end, the decrease in the percentage of the relative error in the non-standard is %82. For fixed-free end and free-free end cases, the decrease in percentages of the relative error is %24.5 and %15.7, respectively, which confirms the lower error in non-standard methods.

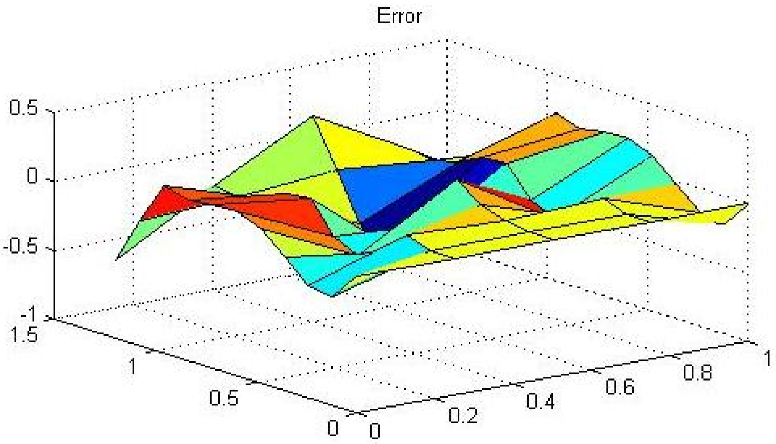

Figure 9 The error of standard method for the for a free-free cable

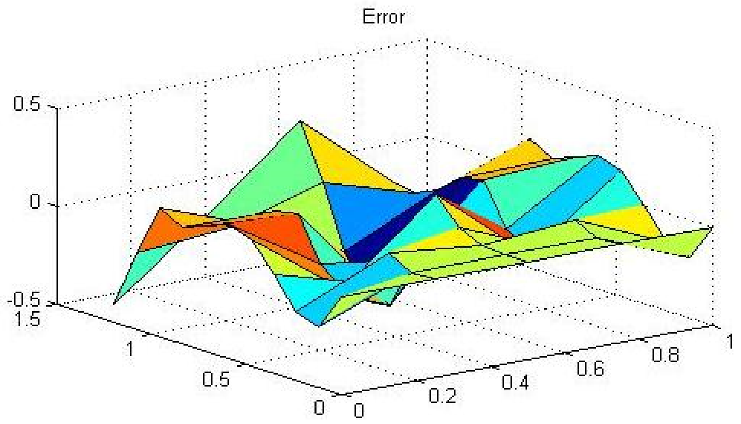

Figure 10 The error of nonstandard method for the free-free cable

Table 1 The comparison of the SSE of NSFD and SFD methods for three BCs of cabl

No.	BCs	Step size and time step	SSE in standard method	SSE in nonstandard method	The percentage of decrease in relative error of the nonstandard
1	Fixed-Fixed end	$\Delta h = 0.02$ $\Delta t = 0.01$	4.6633×10^{-4}	8.9375×10^{-5}	82%
2	Fixed-Free end	$\Delta h = 0.25$ $\Delta t = 0.12$	1.0227	0.7718	24.5%
3	Free-Free end	$\Delta h = 0.25$ $\Delta t = 0.12$	1.8453	1.5538	15.7%

4 Conclusion

In this paper, a cable with three various BCs and sinusoidal initial conditions was considered and solved using three methods of analytical, SFD and NSFD. The results clearly showed the advantage of the nonstandard method compared to the standard method. The error in the nonstandard method was significantly lower than the standard method. At the same time, with the SSE, the error in the nonstandard method was reported between 16 to 82 percent lower than the standard method. According to the lower error in the NSFD method, one can use this powerful method for the solution of some differential equations including the governing equations of cables.

References

[1] Hirota, R., "*The Direct Method in Solution Theory*", Cambridge University Press, (2004).

[2] Olver, P.J., and Rosenau, P., "The Construction of Special Solutions to Partial Differential Equations", Physics Letters A, Vol. 114, pp. 107-112, (1986).

[3] Matinfar, M., and Mahdavi, M ., "Application of Variational Homotopy Perturbation Method for Solving the Cauchy Reaction-diffusion Problem", World Applied Sciences Journal, Vol. 9, pp. 1328-1335, (2010).

[4] Abassy, T.A., El-Tawil, M.A., and El Zoheiry, H., "Toward a Modified Variational Iteration Method", Journal of Computational and Applied Mathematics, Vol. 207, pp. 137-147, (2007).

[5] Ascher, U., "On Symmetric Schemes and Differential-algebraic Equations", SIAM Journal on Scientific and Statistical Computing, Vol. 10, pp. 937-949, (1989).

[6] Butcher, J., and Chan, R., "Efficient Runge-Kutta Integrators for Index-2 Differential Algebraic Equations", Mathematics of Computation of the American Mathematical Society, Vol. 67, pp. 1001-1021, (1998).

[7]Çelik, E., and Bayram, M., "On the Numerical Solution of Differential-algebraic Equations by Padé Series", Applied Mathematics and Computation, Vol. 137, pp. 151-160, (2003).

[8] Hosseini, M.M., "Numerical Solution of Linear Differential-algebraic Equations", Applied Mathematics and Computation, Vol. 162, Issue. 1, pp. 7-14, (2005).

[9] Hosseini, M.M., "An Index Reduction Method for Linear Hessenberg Systems", Applied Mathematics and Computation, Vol. 171, No. 1, pp. 596-603, (2005).

[10] Qureshi, M.I., Hashmi, M., Uddin, Z., and Iqbal, S., "A Study of Obstacle Problems using Homotopy Perturbation Method", Journal of the National Science Foundation of Sri Lanka, Vol. 45, pp. 32-45, (2017).

[11] Somathilake, L., and Wedagedera, J., "Investigation of Branching Structure Formation by Solutions of a Mathematical Model of Pattern Formation in Coral Reefs", Journal of the National Science Foundation of Sri Lanka, Vol. 44, pp. 15-28, (2016).

[12] Majeed, A., Zeeshan, A., and Gorla, R., "Convective Heat Transfer in a Dusty Ferromagnetic Fluid over a Stretching Surface with Prescribed Surface Temperature/heat Flux Including Heat Source/sink", Journal of the National Science Foundation of Sri Lanka, Vol. 46, pp. 56-60, (2018).

[13] Naguleswaran, S., "Transverse Vibration of an Uniform Euler–Bernoulli Beam under Linearly Varying Axial Force", Journal of Sound and Vibration, Vol. 275, pp. 47-57, (2004).

[14] Shakeri, F., Dehghan, M., "Solution of a Model Describing Biological Species Living Together using the Variational Iteration Method", Mathematical and Computer Modelling, Vol. 48, pp. 685-699, (2008).

[15] Biazar, J., Porshokuhi, M.G., and Ghanbari, B., "Extracting a General Iterative Method from an Adomian Decomposition Method and Comparing it to the Variational Iteration Method", Computers & Mathematics with Applications, Vol. 59, pp. 622-628, (2010).

[16] Hsu, J. C., Lai, H. Y., and Chen, C. K., "Free Vibration of Non-uniform Euler–Bernoulli Beams with General Elastically End Constraints using Adomian Modified Decomposition Method", Journal of Sound and Vibration, Vol. 318, pp. 965-981, (2008).

[17]Mickens, R.E., "Calculation of Denominator Functions for Nonstandard Finite Difference Schemes for Differential Equations Satisfying a Positivity Condition", Numerical Methods for Partial Differential Equations: An International Journal, Vol. 23, pp. 672-691, (2007).

[18] Mickens, R.E., and Washington, T.M., "NSFD Discretizations of Interacting Population Models Satisfying Conservation Laws", Computers & Mathematics with Applications, Vol. 66, pp. 2307-2316, (2013).

[19] Roeger, L. I.W., and Mickens, R.E., "Exact Finite-difference Schemes for First Order Differential Equations having Three Distinct Fixed-points", Journal of Difference Equations and Applications, Vol. 13, pp. 1179-1185, (2007).

[20]González-Parra, G., Arenas, A.J., and Chen-Charpentier, B.M., "Combination of Nonstandard Schemes and Richardson's Extrapolation to Improve the Numerical Solution of Population Models", Mathematical and Computer Modelling, Vol. 52, pp. 1030-1036, (2010).

[21] Moaddy, K., Momani, S., and Hashim, I., "The Non-standard Finite Difference Scheme for Linear Fractional PDEs in Fluid Mechanics", Computers & Mathematics with Applications, Vol. 61, pp. 1209-1216, (2011).

[22] Ongun, M.Y., and Turhan, I., "A Numerical Comparison for a Discrete HIV Infection of CD4+ T-Cell Model Derived from Nonstandard Numerical Scheme", Journal of Applied Mathematics, Vol. 3, pp. 61-84, (2012).

[23]Cresson, J., and Pierret, F., "Non Standard Finite Difference Scheme Preserving Dynamical Properties", Journal of Computational and Applied Mathematics, Vol. 303, pp. 15-30, (2016).

[24] Zhang, L., Wang, L., and Ding, X., "Exact Finite Difference Scheme and Nonstandard Finite Difference Scheme for Burgers and Burgers-Fisher Equations", Journal of Applied Mathematics, Vol. 5, pp. 79-92, (2014).

[25] Mickens, R.E., Oyedeji, K., and Rucker, S., "Exact Finite Difference Scheme for Second-order, Linear ODEs Having Constant Coefficients", Journal of Sound and Vibration, Vol. 287, pp. 1052-1056, (2005).

[26] Roeger, L. I.W., "Exact Nonstandard Finite-difference Methods for a Linear System–the Case of Centers", Journal of Difference Equations and Applications, Vol. 14, pp. 381-389, (2008).

[27] Roeger, L. I.W., "Exact Finite-difference Schemes for Two-dimensional Linear Systems with Constant Coefficients", Journal of Computational and Applied Mathematics, Vol. 219, pp. 102-109, (2008).

[28] Zibaei, S., and Namjoo, M., "A Nonstandard Finite Difference Scheme for Solving Fractional-order Model of HIV-1 Infection of CD4^{+} t-cells", Iranian Journal of Mathematical Chemistry, Vol. 6, pp. 169-184, (2015).

Nomenclature

l: length of cable
u: vertical displacement
x: location
t: time
P: tension of cable

Greek Symbol

ω: natural frequency
ρ: density