Spreading of Plastic Zones in Functionally Graded Spherical Tanks Subjected to Internal Pressure and Temperature Gradient Combinations

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Thermo-Elasto-Plastic analyses of thick-walled spherical tanks made of functionally graded materials (FGM) are investigated analytically. These tanks are subjected to positive or negative temperature gradient and internal pressure loadings, separately or simultaneously. The power law modeling has been used for considering the through-thickness variation of mechanical properties. von Mises yield criterion and Elastic-Perfectly-Plastic assumptions are used for describing the material behavior in plastic zones. The patterns of plastic zones spreading for various combinations of internal pressure and positive or negative temperature gradient are investigated. Conducting numerical exercises for similar problems in ABAQUS software shows the excellent agreement with the analytical solutions. FG sphere subjected to internal pressure and negative temperature gradient yields at a lower pressure than the FG sphere subjected to internal pressure. Internal pressure and negative temperature gradient combination is the most critical loading combination. Moreover, the results show that the capacity of FG spherical tank subjected to internal pressure is increased by applying a positive temperature gradient.

Keywords: Analytical analyses, Thermo-Elasto-Plastic stresses, Functionally graded spherical reservoir, Temperature gradient, Internal pressure.

1 Introduction

Functionally Graded Materials (FGMs), was created in (1984) [1]. High strength of FG materials under high temperatures is one of the basic properties of these materials. Publication of papers about FGM has been increased after holding international symposiums of FG materials at Sendai in (1990) [2] and San Francisco in (1992) [3].

Elastic analyses of thick-walled tanks subjected to internal and external pressure carried out by Johnson and Mellor [4] and Cook and Yong[5]. Elasto-Plastic and Thermo-Elasto-Plastic stress analyses of thick-walled spherical tanks have been also done by Whalley [6] and Mendelson [7]. Some studies have been done by Timoshenko and Goodier [8] and Boley and Weiner [9] for homogeneous tanks. Similar researches are also conducted by Nowaki [10].

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Exact solution for Elastic-Perfectly-Plastic behavior of homogeneous sphere under thermal gradient is obtained by Cowper [11].

He also investigated an approximate solution method by neglecting elastic strains which led to conclusion that approximate solution in the case of thermal gradient is almost same as the exact solution. Onset of yielding zones for various temperature-pressure and radius ratios of thick-walled homogeneous spherical tanks has been investigated[12].

Effect of using non homogenous materials in strength and deformation of FG thick-walled cylindrical tanks subjected to internal pressure with assumption of plane strain condition is studied by Fukui and Yamanaka[13]. Fukui et al[14] continued researches with Thermo-mechanical analyses of FG cylinders subjected to temperature gradient. A three dimensional solution was obtained by Vel and Batra [15] for a time-dependent thermal loads imposed to a simply supported functionally graded plate at its top and bottom. Eslami et al.[16] obtained general solution for the one-dimensional steady-state thermal and mechanical stresses in a hollow thick-walled sphere made of FG material.

Shao and Wang [17] solved a steady-state thermal and a non-uniform mechanical loading problem for a FG cylindrical panel with finite length. Thermo-Elastic analysis of a FG cylindrical vessel was performed by Peng and Li[18].

Lack of access to experimental results is led to difficulties in the analytical investigation of FGM's behavior. On the other hand, by increasing industrial demands for such materials, investigation of FGM's behavior is in the focus. There are a few works concentrating on Thermo-Elasto-Plastic analyses of thick-walled FG tanks. The novelty of this paper is to present a comprehensive Thermo-Elasto-Plastic analysis of FG spherical reservoirs by taking into account all possible patterns of plastic zones propagations caused by positive or negative temperature gradients and internal pressure combinations.

In current work, for first time, the analytical Elasto-Plastic and Thermo-Elasto-Plastic analyses of FG spherical thick-walled reservoirs for all possible patterns based on establishing and spreading plastic zones are presented.

2 Governing equations

Inner and outer radiuses of sphere are a and b respectively. Governing equations have been simplified for axisymmetric conditions of geometry, mechanical properties and loading. Across the thickness variation of FG sphere's mechanical properties is determined by power law [19].

$$\begin{aligned} E(r) &= E_0 r^{n_1} \\ \alpha(r) &= \alpha_0 r^{n_2} \\ K(r) &= K_0 r^{n_3} \\ \sigma_y(r) &= \sigma_{y_0} r^{n_4}, \end{aligned} \tag{1}$$

where $E(r)$, $\alpha(r)$, $K(r)$ and $\sigma_y(r)$ are modulus of elasticity, coefficient of linear expansion, thermal conductivity and yield stress, respectively. Also r is radius of sphere and varies between a and b. The parameters E_0, α_0, K_0, σ_{y_0} and n_1 to n_4 are material constants. These parameters are defined in terms of properties and radiuses at inner and outer surfaces of FG sphere as follows:

$$\begin{aligned} E_0 &= \big(E(a) + E(b)\big)/(a^{n_1} + b^{n_1}) \\ \alpha_0 &= \big(\alpha(a) + \alpha(b)\big)/(a^{n_2} + b^{n_2}) \end{aligned} \tag{2}$$

$$K_0 = \left(K(a) + K(b)\right)/(a^{n_3} + b^{n_3})$$

$$\sigma_{y_0} = \left(\sigma_y(a) + \sigma_y(b)\right)/(a^{n_4} + b^{n_4})$$

$$n_1 = \ln(E(b)/E(a))/\ln(b/a)$$

$$n_2 = \ln(\alpha(b)/\alpha(a))/\ln(b/a)$$

$$n_3 = \ln(K(b)/K(a))/\ln(b/a)$$

$$n_4 = \ln\left(\sigma_y(b)/\sigma_y(a)\right)/\ln(b/a).$$

The function "ln" is natural logarithm. The equilibrium and compatibility equations are simplified for symmetrical conditions as follows:

$$\begin{aligned} &\sigma_r'(r) + 2\left(\sigma_r(r) - \sigma_\theta(r)\right)/r = 0 \\ &\varepsilon_\theta'(r) + \left(\varepsilon_\theta(r) - \varepsilon_r(r)\right)/r = 0. \end{aligned} \tag{3}$$

In Eqs. (3), $\sigma_r(r)$ and $\varepsilon_r(r)$ are radial stress and radial strain; also $\sigma_\theta(r)$ and $\varepsilon_\theta(r)$ are hoop stress and circumferential strain respectively. $\sigma_r'(r)$ and $\varepsilon_\theta'(r)$ are first derivative of $\sigma_r(r)$ and $\varepsilon_\theta(r)$ with respect to r, respectively. Hook's law for FG sphere is modified in Eqs. (4).

$$\begin{aligned} &\varepsilon_r(r) = \alpha_0 r^{n_2} T(r) + \left(\sigma_r^e(r) - 2\nu\sigma_\theta^e(r)\right)/E_0 r^{n_1} \\ &\varepsilon_\theta(r) = \alpha_0 r^{n_2} T(r) + \left((1-\nu)\sigma_\theta^e(r) - \nu\sigma_r^e(r)\right)/E_0 r^{n_1}, \end{aligned} \tag{4}$$

where $\sigma_r^e(r)$ and $\sigma_\theta^e(r)$ are elastic radial and circumferential stresses respectively and $T(r)$ is temperature at radius r. One dimensional differential equation of steady-state heat conduction is written in Eq. (5).

$$rK(r)T''(r) + \left(rK'(r) + 2K(r)\right)T'(r) = 0 \tag{5}$$

where $K'(r)$ is first derivative and $T'(r)$ and $T''(r)$ are first and second derivatives of $K(r)$ and $T(r)$ with respect to r, respectively. Tresca and von Mises yield criteria for FG sphere under axisymmetric conditions have been simplified as follows:

$$|\sigma_{eff}(r)| = |\sigma_\theta(r) - \sigma_r(r)| = \sigma_y(r), \tag{6}$$

in which $\sigma_{eff}(r)$ denotes the effective stress.

3 Formulation

Difference between temperatures at inner and outer radiuses of FG sphere is assumed equal to ΔT. Solution of differential equation in Eq. (5) is written in Eq. (7).

$$T(r) = \Delta T\left(r^{-(n_3+1)} - b^{-(n_3+1)}\right)/\left(a^{-(n_3+1)} - b^{-(n_3+1)}\right) \tag{7}$$

In FG sphere subjected to positive temperature gradient, the temperature of inner surface is greater than the temperature of outer surface (i.e. $T(a) > T(b)$ or $\Delta T > 0$) and vice versa. Governing differential equation of plastic zone is obtained by combining equilibrium equation and yield criterion.

$$\sigma_r'(r) - 2\sigma_{y_0} r^{n_4-1} = 0. \tag{8}$$

Considering Eq. (6), and conducting numerical exercises for all possible thermal gradient and internal pressure combinations, implies that the maximum absolute value of effective stress is at inner or outer radii, therefore only two plastic zones can be established separately or simultaneously, in general. These plastic zones are illustrated in Figure (1). Inner and outer radiuses of elastic zones are assumed equal to $r_1 = c$ and $r_2 = d$, respectively. In general r_1 is equal to a or c and r_2 is equal to d or b. Governing differential equation of elastic zone is obtained by combining equilibrium and compatibility equations by considering Hook's law and Eq. (7) as follows:

$$r^2(1-\nu)\sigma_r''(r) + r(4-n_1)(1-\nu)\sigma_r'(r) - 2n_1(1-2\nu)\sigma_r(r) = 2E_0 r^{n_1+1} H(r), \tag{9}$$

in which $H(r)$ is defined in Eq. (10).The constant R is equal to r_2/r_1.

$$H(r) = \alpha_0 \Delta T\left(r^{n_2-n_3-2}(n_3-n_2+1)r_2^{n_3+1} + n_2 r^{n_2-1}\right)/(R^{n_3+1}-1) \tag{10}$$

Solution of Eq. (8) for first and second plastic zones is presented in Eqs. (11).

$$\begin{gathered}\sigma_r^{p1}(r) = \varphi_1(r) - P \\ \sigma_r^{p2}(r) = \varphi_2(r)\end{gathered} \tag{11}$$

where P is internal pressure, and $\varphi_1(r)$ and $\varphi_2(r)$ are as follows:

$$\begin{gathered}\varphi_1(r) = \pm 2\sigma_{y_0}(r^{n_4} - a^{n_4})/n_4 \\ \varphi_2(r) = \pm 2\sigma_{y_0}(b^{n_4} - r^{n_4})/n_4 \,.\end{gathered} \tag{12}$$

$$\alpha_0 \Delta T\left(r^{n_2-n_3-2}(n_3-n_2+1)r_2^{n_3+1} + n_2 r^{n_2-1}\right)/(R^{n_3+1}-1)$$

Plastic circumferential stresses for first and second plastic zones (i.e. $\sigma_\theta^{p1}(r)$ and $\sigma_\theta^{p2}(r)$) are obtained by using equilibrium equation.

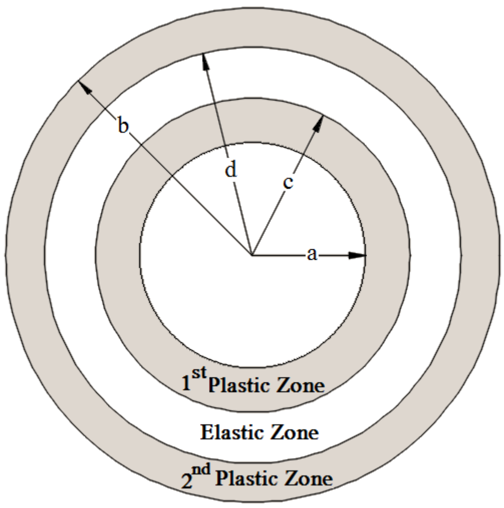

Figure 1 Elastic and plastic zones

$$\sigma_\theta^{p1}(r) = \vartheta_1(r) - P$$
$$\sigma_\theta^{p2}(r) = \vartheta_2(r) \tag{13}$$

The functions $\vartheta_1(r)$ and $\vartheta_2(r)$ in Eqs. (13), are defined in Eqs. (14).

$$\vartheta_1(r) = \pm\,\sigma_{y_0}(r^{n_4}(2+n_4) - 2a^{n_4})/n_4$$
$$\vartheta_2(r) = \pm\,\sigma_{y_0}\big(2b^{n_4} - r^{n_4}(2+n_4)\big)/n_4 \tag{14}$$

Plus or minus sign of $\varphi_1(r)$, $\varphi_2(r)$, $\vartheta_1(r)$ and $\vartheta_2(r)$ are depended on sign of effective stress. For the case that FG spherical reservoir subjected to internal pressure (i.e. ΔT is ignored) plus sign is used. For the case that FG sphere is subjected to temperature gradient (i.e. P is neglected) or the case that the tank is subjected to internal pressure and temperature gradient simultaneously, the signs of Eqs. (12) and (14) and the sign of ΔT are opposite. Solution of differential equation in Eq. (9) is written in Eq. (15).

$$\sigma_r^e(r) = \xi(r,r_2)P + Z(r,r_1,r_2)\Delta T + \Phi(r,r_1,r_2), \tag{15}$$

Where $\xi(r,r_2)$, $Z(r,r_1,r_2)$ and $\Phi(r,r_1,r_2)$ are defined in Eqs. (16).

$$\xi(r,r_2) = \sum_{i=1}^{2}\frac{(-1)^i r_2^{\lambda_{3-i}} r^{\lambda_i}}{\beta(r_1,r_2)}$$
$$Z(r,r_1,r_2) = \zeta(r,r_1,r_2) - \sum_{j=1}^{2}\sum_{i=1}^{2}\frac{(-1)^{i+j}\zeta(r_i,r_1,r_2) r_{3-i}^{\lambda_{3-j}} r^{\lambda_j}}{\beta(r_1,r_2)} \tag{16}$$
$$\Phi(r,r_1,r_2) = \sum_{i=1}^{2}\frac{(-1)^i \phi_i(r_1,r_2) r^{\lambda_i}}{\beta(r_1,r_2)}$$

In Eqs. (16), $\beta(r_1,r_2)$, $\zeta(r,r_1,r_2)$, $\phi_i(r_1,r_2)$ and λ_i are presented in Eqs. (17).

$$\beta(r_1,r_2) = r_1^{\lambda_1} r_2^{\lambda_2} - r_2^{\lambda_1} r_1^{\lambda_2}$$
$$\zeta(r,r_1,r_2) = h(r,r_1,r_2)\left(r_1^{-(1+n_3)} - r_2^{-(1+n_3)}\right)/\left(a^{-(1+n_3)} - b^{-(1+n_3)}\right)$$
$$\phi_i(r_1,r_2) = \sum_{j=1}^{2}(-1)^j \varphi_j(r_j) r_{3-j}^{\lambda_{3-i}} \tag{17}$$
$$\lambda_i = \big((1-\nu)(n_1-3) + (-1)^{i+1}C\big)/\big(2(1-\nu)\big)$$

The function $h(r,r_1,r_2)$ is presented in Eq. (18).

$$h(r,r_1,r_2) = \frac{4E_0\alpha_0(\nu-1)\big(A(n_3-n_2+1)r_2^{n_3+1} r^{n_1+n_2-n_3-1} + n_2 B r^{n_1+n_2}\big)}{C(R^{n_3+1}-1)}. \tag{18}$$

The constants A, B and C are determined in Eqs. (19).

$$\begin{gathered} A=\left((\nu-1)(2n_2+n_1+1)+C\right)^{-1}-\left((\nu-1)(2n_2+n_1+1)-C\right)^{-1} \\ B=\left((\nu-1)(2n_2+n_1+3)+C\right)^{-1}-\left((\nu-1)(2n_2+n_1+3)-C\right)^{-1} \\ C=\sqrt{(1-\nu)\left((n_1-3)^2(1-\nu)+8n_1(1-2\nu)\right)} \end{gathered} \tag{19}$$

The elastic tangential stress after using equilibrium equation is obtained.

$$\sigma_\theta^e(r)=\left(\xi(r,r_2)+\bar{\xi}(r,r_2)\right)P+\left(Z(r,r_1,r_2)+\bar{Z}(r,r_1,r_2)\right)\Delta T+\Phi(r,r_1,r_2)+\bar{\Phi}(r,r_1,r_2)\,. \tag{20}$$

where $\bar{\xi}(r,r_2)$, $\bar{Z}(r,r_1,r_2)$ and $\bar{\Phi}(r,r_1,r_2)$ are as follows:

$$\begin{gathered} \bar{\xi}(r,r_2)=r(d/dr)\left(\xi(r,r_2)\right)/2 \\ \bar{Z}(r,r_1,r_2)=r(d/dr)\left(Z(r,r_1,r_2)\right)/2 \\ \bar{\Phi}(r,r_1,r_2)=r(d/dr)\left(\Phi(r,r_1,r_2)\right)/2\,. \end{gathered} \tag{21}$$

4 Patterns of plastic zones spreading

For all possible combinations of internal pressure and positive or negative temperature gradient, the patterns of plastic zones propagation in FG spherical reservoirs have been investigated.

4-1 FG sphere subjected to internal pressure

For this case, sign of effective stress at any radius of elastic FG sphere is invariant. Moreover the maximum absolute amount of effective stress is occurred at $r = a$. Therefore elastic FG sphere initiates yielding from inner surface. Yield pressure after satisfying Eq. (6) at $r = a$ has been obtained as follows:

$$P_y=\sigma_{y_0}a^{n_4}/\bar{\xi}(a,b). \tag{22}$$

By increasing P from P_y to P_c, first plastic zone is propagated from $r = a$ to $r_1 = c$. The corresponding internal pressure is calculated by satisfying Eq. (6) at $r = c$.

$$P_c=\left(\sigma_{y_0}c^{n_4}-\bar{\Phi}(c,c,b)\right)/\bar{\xi}(c,b) \tag{23}$$

The first plastic zone is established without establishing the second plastic zone (i.e. $r_2 = b$ in Figure (1)). In limit state, by approaching r_1 from c to b, the capacity of FG sphere is obtained as follows:

$$P_{max}=\lim_{c\to b}P_c=\varphi_1(b)\,. \tag{24}$$

In addition to, after approaching n_4 to zero, the well-known formula of homogeneous spherical tank's capacity is obtained and the validity of Eq. (24) is observed.

$$P_{max} = \lim_{n_4 \to 0} \varphi_1(b) = 2\sigma_{y_0} \ln(b/a) \tag{25}$$

4-2 FG sphere subjected to temperature gradient

For this case, signs of effective stresses in elastic FG sphere at r = a and r = b are opposite to each other. Maximum absolute amount of effective stress for positive or negative temperature gradient has been occurred at r = a. Hence, elastic FG sphere initiates yielding from inner surface. Yield temperature gradient after satisfying Eq. (6) at r = a is obtained as follows (for negative temperature gradient i.e. $\Delta T < 0$, the positive sign is used and vice versa):

$$T_y = \pm\, \sigma_{y_0} a^{n_4} / \bar{Z}(a, a, b). \tag{26}$$

By increasing ΔT from T_y for positive temperature gradient or decreasing ΔT from T_y for negative temperature gradient, first plastic zone is propagated from r = a to $r_1 = c$. When ΔT is equal to $T_{y,b}$ the second plastic zone is established. The corresponding first plastic zone's outer radius (i.e. $r_1 = c_0$) and $T_{y,b}$ for positive or negative temperature gradients are obtained by satisfying Eq. (6) at $r = c_0$ and r = b. The radius c_0 is obtained by solving Eq. (27) with respect to c_0.

$$\psi(c_0) + \psi(b) = 0 \tag{27}$$

where $\psi(r)$ is as follows:

$$\psi(r) = \left(\sigma_{y_0} r^{n_4} + k\bar{\Phi}(r, c_0, b)\right) / \bar{Z}(r, c_0, b) \tag{28}$$

The parameter k for $r = c_0$ is equal to -1 and for r = b is equal to +1 when ΔT is negative and vice versa. After obtaining c_0 from Eq. (27), $T_{y,b}$ is obtained as follows (positive sign is for $\Delta T < 0$ and vice versa):

$$T_{y,b} = \pm\psi(c_0). \tag{29}$$

By increasing ΔT from $T_{y,b}$ for positive temperature gradient or decreasing ΔT from $T_{y,b}$ for negative temperature gradient, the effective stress at a radius between c and d in Figure (1) is vanished and the capacity of FG sphere is approached to infinity. Therefore, using yield criterion based on maximum shear energy shows unreal capacity for reservoirs subjected to temperature gradient.

4-3 FG sphere subjected to internal pressure and temperature gradient

In elastic or Elasto-Plastic FG sphere, the signs of effective stresses for positive and negative temperature gradients (P = 0) are opposite to each other. Also, temperature gradient produces effective stresses with different signs at r = a and r = b, as mentioned before. Because of these differences, FG sphere must be studied for positive and negative temperature gradients separately. Elastic stress field for each case can be obtained by substituting r_1 and r_2 by corresponding radii of first and second plastic zones in Eqs. (15) to (18) and Eqs. (20) and (21).

4-3-1 Internal pressure and positive temperature gradient

Elastic FG sphere subjected to internal pressure ($\Delta T = 0$) or subjected to positive temperature gradient ($P = 0$) yields at $r = a$. Sign of effective stress caused by positive temperature gradient is opposite to the sign of effective stress caused by internal pressure at $r = a$. Thus, this load combination increases the capacity of elastic FG sphere. For describing possible patterns of plastic zones spreading, the specific amounts of internal pressure (i.e. $P^{PTG}(r_1, r_2)$) and positive temperature gradient (i.e. $T^{PTG}(r_1, r_2)$) are introduced in Eqs. (30). These parameters have been obtained by satisfying Eq. (6) at $r = r_1$ and $r = r_2$.

$$\begin{aligned} P^{PTG}(r_1, r_2) &= \left(\mu_{\bar{Z}}(r_1, r_2) + \sigma_{y_0}\eta^{+}_{\bar{Z}}(r_1, r_2)\right)/\gamma(r_1, r_2) \\ T^{PTG}(r_1, r_2) &= \left(\mu_{\bar{\xi}}(r_1, r_2) - \sigma_{y_0}\eta^{+}_{\bar{\xi}}(r_1, r_2)\right)/\gamma(r_1, r_2) \end{aligned} \tag{30}$$

where $\gamma(r_1, r_2)$, $\mu_{\bar{Z}}(r_1, r_2)$, $\eta^{+}_{\bar{Z}}(r_1, r_2)$, $\mu_{\bar{\xi}}(r_1, r_2)$ and $\eta^{+}_{\bar{\xi}}(r_1, r_2)$ are as follows:

$$\begin{aligned} \gamma(r_1, r_2) &= \bar{\xi}(r_2, r_2)\bar{Z}(r_1, r_1, r_2) - \bar{\xi}(r_1, r_2)\bar{Z}(r_2, r_1, r_2) \\ \mu_{\bar{Z}}(r_1, r_2) &= \bar{\Phi}(r_1, r_1, r_2)\bar{Z}(r_2, r_1, r_2) - \bar{\Phi}(r_2, r_1, r_2)\bar{Z}(r_1, r_1, r_2) \\ \eta^{+}_{\bar{Z}}(r_1, r_2) &= \bar{Z}(r_2, r_1, r_2)r_1^{n_4} + \bar{Z}(r_1, r_1, r_2)r_2^{n_4} \\ \mu_{\bar{\xi}}(r_1, r_2) &= \bar{\xi}(r_1, r_2)\bar{\Phi}(r_2, r_1, r_2) - \bar{\xi}(r_2, r_2)\bar{\Phi}(r_1, r_1, r_2) \\ \eta^{+}_{\bar{\xi}}(r_1, r_2) &= \bar{\xi}(r_2, r_2)r_1^{n_4} + \bar{\xi}(r_1, r_2)r_2^{n_4}. \end{aligned} \tag{31}$$

4-3-1-1 Onset of yielding from inner surface

For this case, elastic FG sphere is subjected to internal pressure and positive temperature gradient less than $P^{PTG}(a, b)$ and $T^{PTG}(a, b)$ respectively. By increasing ΔT and decreasing P first plastic zone has been established. Elasto-Plastic FG sphere is subjected to internal pressure and positive temperature gradient less than $P^{PTG}(c, b)$ and $T^{PTG}(c, b)$, respectively. By increasing r_1 from radius a to specific radius equal to c, internal pressure has been vanished. This specific radius is obtained by solving Eq. (32) with respect to c. This radius is equal to c_0 in Eq. (27).

$$P^{PTG}(c, b) = 0 \tag{32}$$

In this pattern FG sphere after vanishing P, is subjected to positive temperature gradient equal to Eq. (33) and second plastic zone is established.

$$T^{PTG}(c_0, b) = -\psi(c_0) \tag{33}$$

4-3-1-2 Onset of yielding from outer surface

The elastic FG sphere is subjected to internal pressure less than $P^{PTG}(a, b)$ and positive temperature gradient greater than $T^{PTG}(a, b)$. In this pattern, radius r_1 is invariant ($r_1 = a$) and radius r_2 is equal to d. The radius r_2 varies from b toward a. Therefore second plastic zone spreads without establishing first plastic zone.

The Elasto-Plastic FG sphere is subjected to internal pressure less than $P^{PTG}(a,d)$ and positive temperature gradient greater than $T^{PTG}(a,d)$. By approaching radius d to radius a, the specific internal pressure $P^{PTG}(a,a)$ and specific positive temperature gradient $T^{PTG}(a,a)$ are approached to infinity.

4-3-1-3 Onset of yielding from inner and outer surfaces

Elastic FG sphere is subjected to internal pressure and positive temperature gradient equal to $P^{PTG}(a,b)$ and $T^{PTG}(a,b)$ respectively. First and second plastic zones are established simultaneously. In addition to, Elasto-Plastic FG sphere is subjected to internal pressure and positive temperature gradient equal to $P^{PTG}(c,d)$ and $T^{PTG}(c,d)$ respectively.

4-3-2 Internal pressure and negative temperature gradient

Elastic FG sphere subjected to internal pressure ($\Delta T = 0$) or subjected to negative temperature gradient ($P = 0$) yields at $r = a$. The sign of effective stress caused by negative temperature gradient is similar to the sign of effective stress caused by internal pressure at $r = a$. Therefore this combination decreases the capacity of elastic FG sphere and is the critical load combination. In this case, internal pressure for yielding of elastic FG sphere is less than P_y in Eq. (22). After satisfying Eq. (6), the reduction of capacity is obtained as follows:

$$P_y - P = \Delta T \bar{\bar{Z}}(a,a,b)/\bar{\bar{\xi}}(a,b). \tag{34}$$

For describing possible patterns of plastic zones spreading, the specific amounts of internal pressure (i.e. $P^{NTG}(r_1)$) and negative temperature gradient (i.e. $T^{NTG}(r_1)$) are introduced in Eqs. (35). These parameters are obtainde by satisfying Eq. (6) at $r = r_1$ and the equation $\sigma_{eff}(b) = 0$.

$$\begin{aligned} P^{NTG}(r_1) &= \left(\mu_{\bar{Z}}(r_1,b) - \sigma_{y_0}\eta_{\bar{\bar{Z}}}(r_1)\right)/\gamma(r_1,b) \\ T^{NTG}(r_1) &= \left(\mu_{\bar{\xi}}(r_1,b) + \sigma_{y_0}\eta_{\bar{\bar{\xi}}}(r_1)\right)/\gamma(r_1,b) \end{aligned} \tag{35}$$

in which $\eta_{\bar{\bar{Z}}}(r_1)$ and $\eta_{\bar{\bar{\xi}}}(r_1)$ are as follows:

$$\begin{aligned} \eta_{\bar{\bar{Z}}}(r_1) &= \bar{\bar{Z}}(b,r_1,b)r_1^{n_4} \\ \eta_{\bar{\bar{\xi}}}(r_1) &= \bar{\bar{\xi}}(b,b)r_1^{n_4}. \end{aligned} \tag{36}$$

4-3-2-1 Establishing of first plastic zone

For this case, elastic FG sphere is subjected to internal pressure less than $P^{NTG}(a)$ and $|\Delta T| > |T^{NTG}(a)|$. In this pattern the radius r_2 is invariant and radius r_1 is equal to c. Meanwhile, the radius r_1 varies from radius a to radius b (see Figure (1)). First plastic zone spreads without establishing second plastic zone. Also the Elasto-Plastic FG sphere is subjected to internal pressure and negative temperature gradient less than $P^{NTG}(c)$ and $T^{NTG}(c)$, respectively.
By approaching radius c to radius b the specific internal pressure $P^{NTG}(b)$ and specific negative temperature gradient $T^{NTG}(b)$ are approached to positive and negative infinity respectively.

4-3-2-2 Establishing of first and second plastic zones

Order of establishing plastic zones are similar to the pattern that mentioned in subsection 4-3-1-1; but combination of internal pressure and temperature gradient in this pattern has been occurred in a vice versa manner. In this pattern, first plastic zone is established by applying negative temperature gradient equal to Eq. (26) without internal pressure. On threshold of establishing second plastic zone, negative temperature gradient is equal to Eq. (29), where c_0 is obtained from Eq. (27). After forming second plastic zone, the FG sphere bears both the internal pressure and the negative temperature gradient simultaneously.

5 Results and discussion

Elasto-Plastic and Thermo-Elasto-Plastic analyses of thick-walled spherical tanks made of functionally graded materials are conducted analytically. For perusing result validation, a section of FG sphere composed of homogeneous layers having different mechanical properties from each other's is modeled in ABAQUS software. The modulus of elasticity, coefficient of linear expansion and yield stress of homogeneous layers are calculated from Eqs. (1) at centroid of each layer. The supports are defined in local spherical coordinate system. Pressure is applied to the internal face of the model as a pressure mechanical load with a uniform distribution. Temperature gradient is defined as a boundary condition with radial distribution through the model's thickness. Variations of Young modulus and yield stress in radial direction for a certain specimen of FG material made of Al/SiCp is obtained by Rodrı´guez-Castro et al.[20]. Parvizi and Naghdabadi [21] calibrated these experimental data for FG sphere with inner and outer radiuses equal to 0.4 m and 0.8 m. They also obtained thermal expansion coefficient and thermal conductivity by using volume fraction law. The constants of Eqs. (1) for special case, the FG spherical reservoir made of Al/SiCp with 0.3 volume fraction of SiC at inner surface and 0.2 volume fraction of SiC at outer surface, are presented in Table (1).
We used these constants for Elasto-Plastic analyses of FG spherical reservoirs and cylindrical vessels subjected to internal pressure [22] and Thermo-Elastio-Plastic analysis of spherical tanks subjected to temperature gradient [23] in previous studies. Moreover, some researches are conducted for stability analyses of FG beam and plates [24-27].
In Figure (2) homogeneous and FG spheres are subjected to yield pressure ($T = 0$). The yield stress of metallic tank is equal to 80 MPa. This figure indicates that capacity of FG sphere is higher than capacity of homogeneous metallic sphere. Figure (3) illustrates the Elasto-Plastic stress field of FG sphere ($T = 0$). In Figure (3), dimensionless parameter R_1 is $(c-a)/(b-a)$, where $a \leq c \leq b$.
The subscript 1 in R_1 denotes first plastic zone. In Figure (4) FG sphere is subjected to positive yield temperature gradient ($P = 0$). In Figure (5) Elasto-Plastic stress field of FG sphere subjected to positive temperature gradient is presented. In this figure parameter R_1 is equal to $(c-a)/(c_0-a)$ for $a \leq c \leq c_0$. In Figure (6) FG sphere is subjected to specific internal pressure and positive temperature gradient for $r_1 = a$ and $r_2 = b$. In Figure (7) Elasto-Plastic stress field of FG sphere subjected to specific internal pressure and positive temperature gradient is presented. The dimensionless parameters A and B are equal to $\Delta T/T^{PTG}(c, b)$ and $P/P^{PTG}(c, b)$ respectively. The radius c_0 is equal to 466.483 mm.
In this figure parameter R_1 is equal to $(c-a)/(c_0-a)$, in which $a \leq c \leq c_0$. In Figure (8), FG sphere is subjected to specific positive temperature gradient and constant specific internal pressure. Figures (7) and (8) are presented for pattern that mentioned in subsection 4-3-1-1. Also the Figure (9) is presented for pattern that mentioned in subsection 4-3-1-3. In this figure dimensionless parameters A and B are equal to $\Delta T/T^{PTG}(c, d)$ and $P/P^{PTG}(c, d)$, respectively.

Table 1 Material constants of FG sphere made of Al/SiCp [21]

$a(m)$	$b(m)$	ν	n_1	n_2	n_3	n_4	α_0	$E_0(GPa)$	$\sigma_{y_0}(MPa)$
0.4	0.8	0.3	-0.43	0.133	0.124	-0.41	18.03×10^{-6}	34.4	85.8

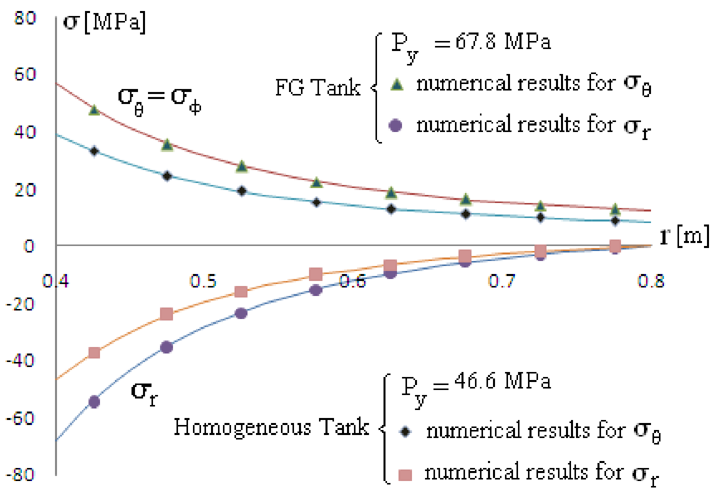

Figure 2 Homogeneous and FG Sphere subjected to internal yield pressure

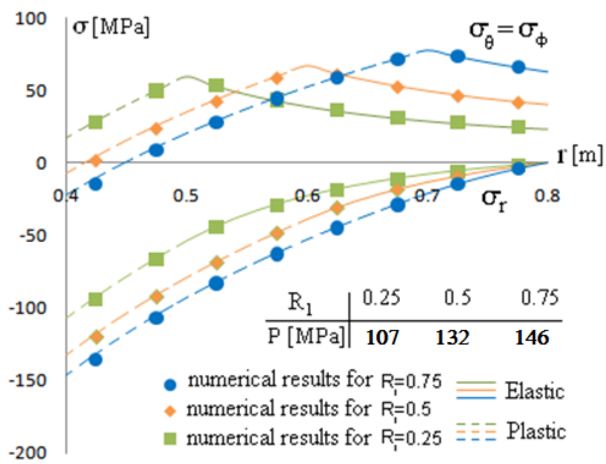

Figure 3 Elasto-Plastic FG Sphere subjected to internal pressure

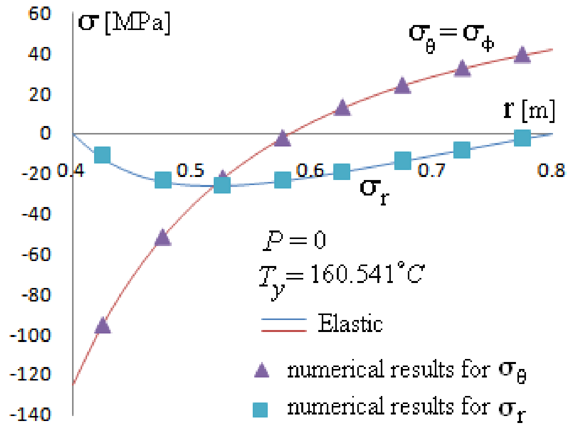

Figure 4 FG Sphere subjected to positive yield temperature gradient

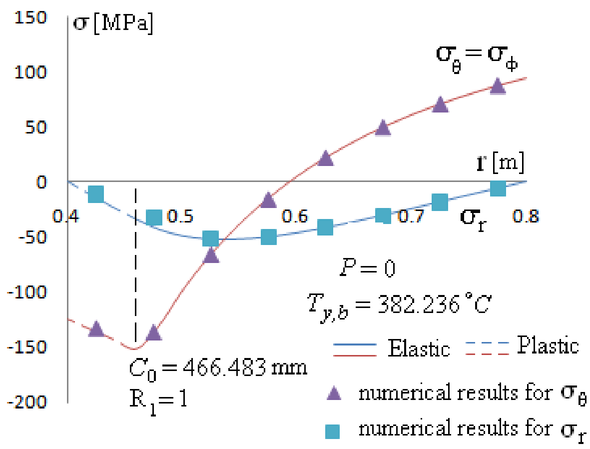

Figure 5 Elasto-Plastic stress field of FG sphere subjected to positive temperature gradient

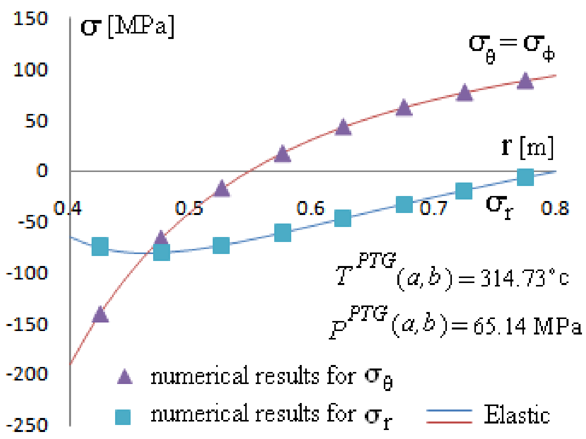

Figure 6 FG Sphere subjected to specific internal pressure and positive temperature gradient

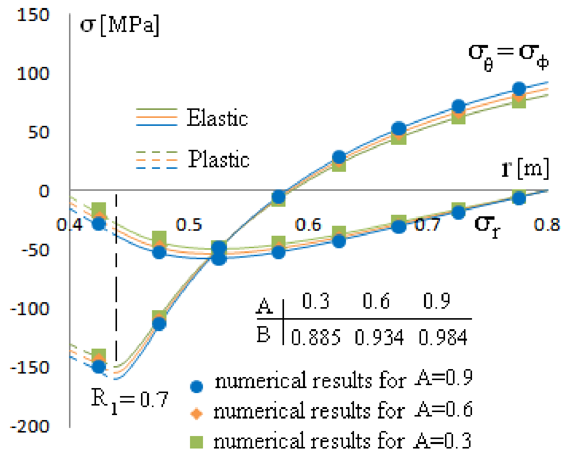

Figure 7 Elasto-Plastic stress field of FG tank subjected to specific internal pressure and positive temperature gradient

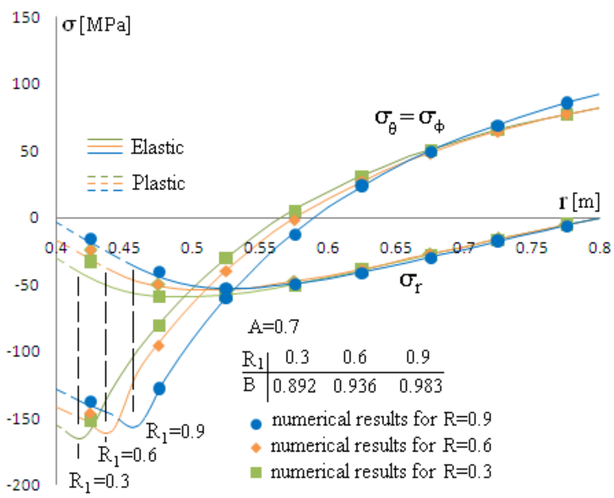

Figure 8 Elasto-Plastic stress field of FG sphere subjected to specific positive temperature gradient and constant internal pressure

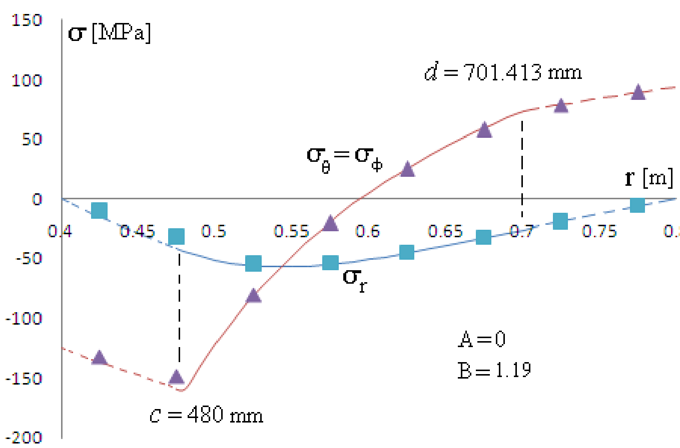

Figure 9 Elasto-Plastic stress field after vanishing P in first pattern

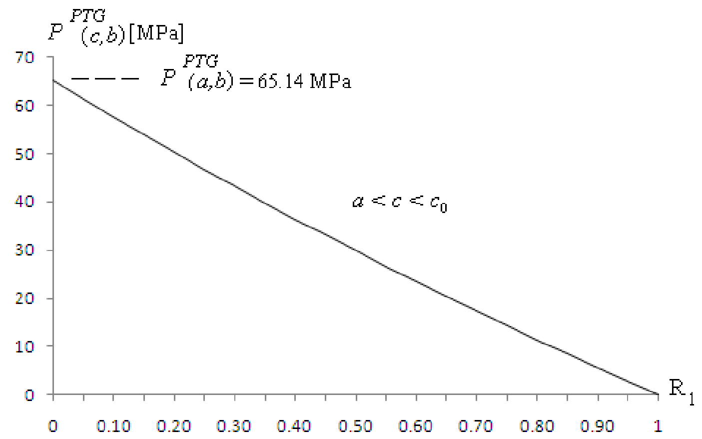

Figure 10 Specific internal pressure for first pattern

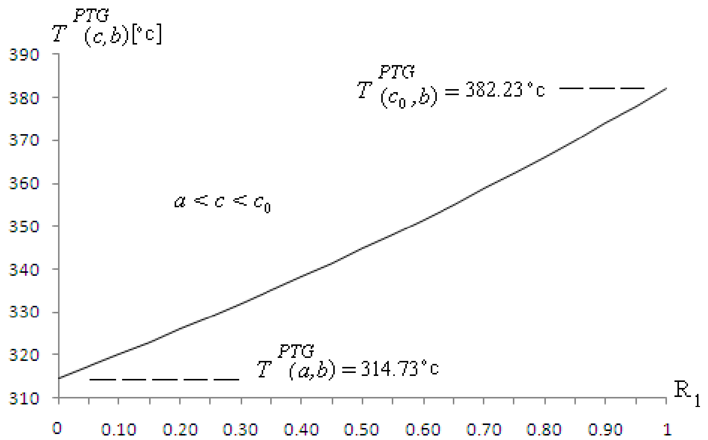

Figure 11 Specific positive temperature gradient for first pattern

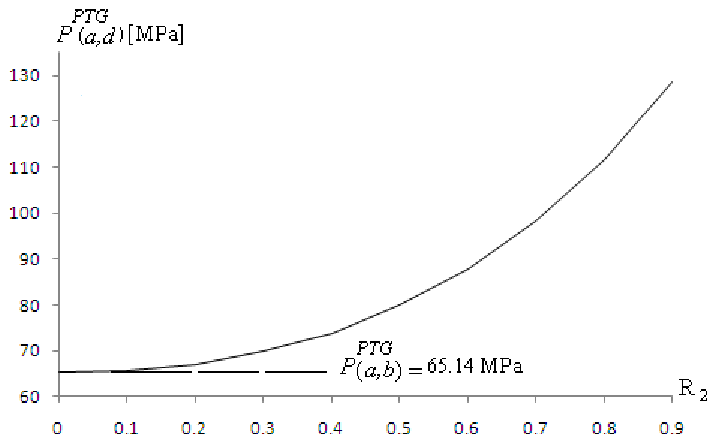

Figure 12 Specific internal pressure for second pattern

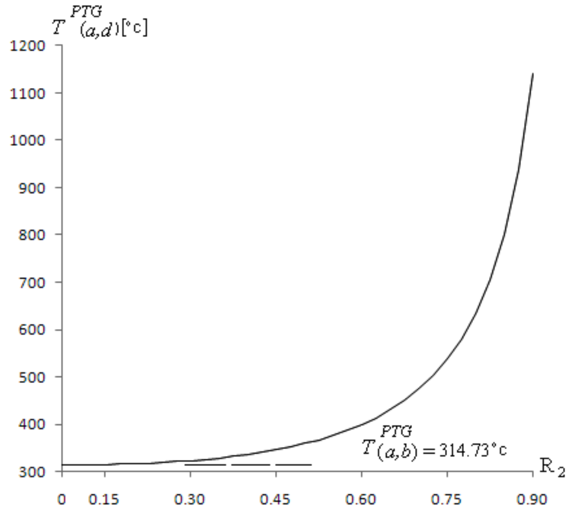

Figure 13 Specific positive temperature gradient for second pattern

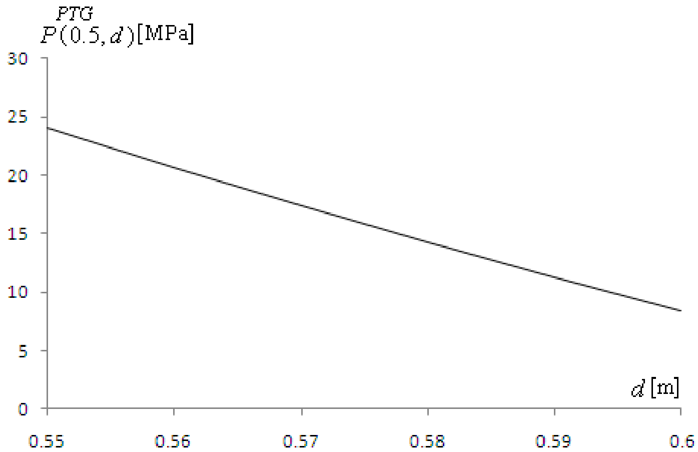

Figure 14 Specific internal pressure for third pattern

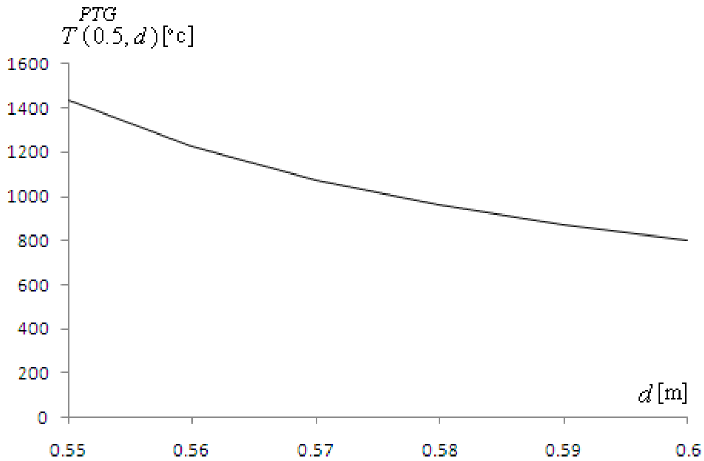

Figure 15 Specific positive temperature gradient for third pattern

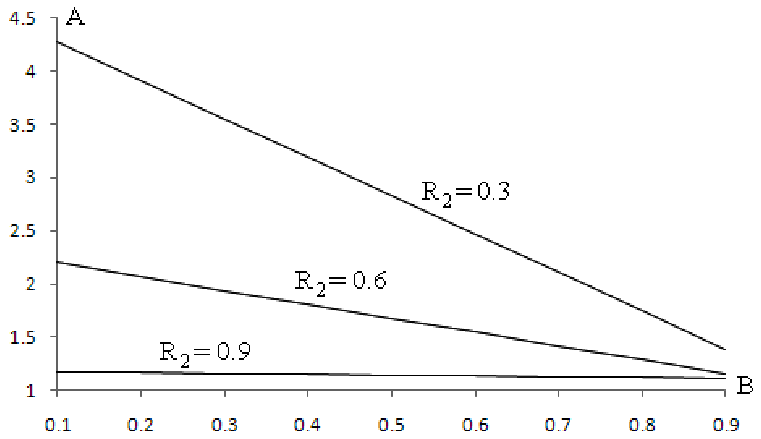

Figure 16 Dimensionless diagram for second pattern of internal pressure and positive temperature gradient

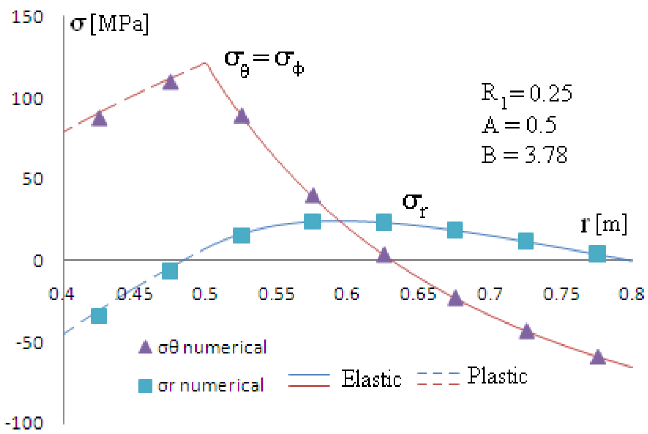

Figure 17 Elasto-Plastic stress field of FG sphere subjected to specific negative temperature gradient and internal pressure

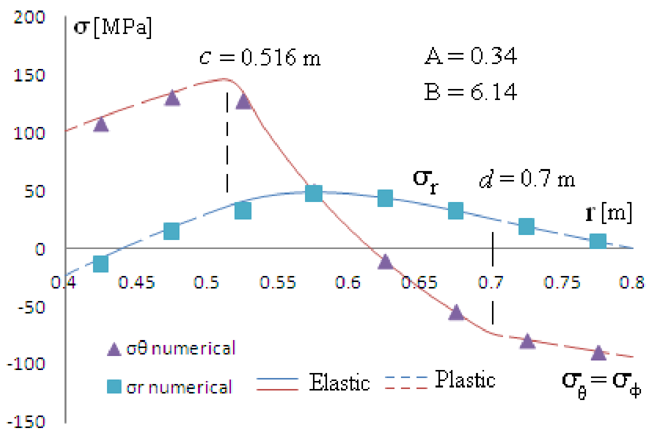

Figure 18 Elasto-Plastic stress field for second pattern of internal pressure and negative temperature gradient combination

In Figures (10) to (15), specific internal pressure and specific positive temperature gradient for first, second and third patterns are presented.

In Figure (16) dimensionless diagrams for second pattern of internal pressure and positive temperature gradient combination is presented. In this figure parameters A and B are equal to $\Delta T/T^{PTG}(a, d)$ and $P/P^{PTG}(a, d)$ respectively.

The dimensionless parameter R_2 is equal to $(b-d)/(b-a)$, where $a \leq d \leq b$. The subscript 2 in the symbol R_2 denotes second plastic zone. Elasto-Plastic stress field of FG sphere for the first pattern of internal pressure and negative temperature gradient combination is illustrated in Figure (17), where the dimensionless parameters A and B are equal to $\Delta T/T^{NTG}(c)$ and $P/P^{NTG}(c)$ respectively. In this figure parameter R_1 is equal to $(c-a)/(b-a)$ and $a \leq c \leq b$. Elasto-Plastic stress field of FG sphere for second pattern of internal pressure and negative temperature gradient combination is illustrated in Figure (18).

6 Conclusion

In current work a comprehensive Elasto-Plastic and Thermo-Elasto-Plastic analyses of FG spherical reservoirs by taking into account all possible patterns of plastic zones spreading due to positive or negative temperature gradients and internal pressure combinations, is presented. The behavior of material in plastic zones is described by Elastic-Perfectly-Plastic model and von Misses yield criterion. The analytical solutions are verified by conducting numerical exercises in ABAQUS software. These results show that:

- Plastic zone in homogeneous and FG spherical tanks subjected to internal pressure initiates from inner radius and spreads to outer radius uniformly.
- In FG spherical tank subjected to positive or negative temperature gradient, first plastic zone initiates from inner radius and spreads to a radius less than outer radius uniformly; then the tank yields from outer radius and second plastic zone is established.
- In FG spherical tank subjected to internal pressure and positive temperature gradient, the first or second plastic zones can be formed simultaneously or separately. Therefore for this case three possible patterns of plastic zone propagation can be formed.
- In FG spherical reservoir subjected to internal pressure and negative temperature gradient, first plastic zone propagates toward outer radius uniformly or second plastic zone initiates at the outer radius. In this case, two possible patterns of spreading plastic zones have been observed.
- FG sphere subjected to internal pressure and positive temperature gradient yields at a higher pressure than FG sphere subjected to internal pressure. FG sphere subjected to internal pressure and negative temperature gradient yields at a lower pressure than the FG sphere subjected to internal pressure. Therefore internal pressure and negative temperature gradient combination is the most critical combination. Also the positive temperature gradient increases the capacity of FG sphere subjected to internal pressure.
- Using a section of laminated sphere for finite element modeling of axisymmetric FG sphere, yields to an accurate Thermo-Elasto-plastic analysis.
- Considering a model with the more layers, results the more accuracy. However, for FG spherical tank with the properties that presented in Table (1), using eight homogeneous layers and assigning to each layer the properties of FG material at midpoint, leads to an accurate result.

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Nomenclature

a = inner radius
b = outer radius
$E(r)$ = modulus of elasticity at radius r
E_0 = material constant for modulus of elasticity
$K(r)$ = thermal conductivity at radius r
K_0 = material constant for thermal conductivity

n_1 = material constant for modulus of elasticity in power law distribution
n_2 = material constant for linear expansion coefficient in power law distribution
n_3 = material constant for thermal conductivity in power law distribution
n_4 = material constant for yield stress in power law distribution
P = internal pressure
P_c = pressure corresponding to first plastic zone propagation
P_y = yield pressure in reservoir subjected to internal pressure
$P^{NTG}(r_1)$ = specific pressure corresponding to negative temperature gradient
$P^{PTG}(r_1, r_2)$ = specific pressure corresponding to positive temperature gradient
r_1= boundary radius between elastic and first plastic zones
r_2= boundary radius between elastic and second plastic zones
R_1 = the ratio of first plastic zone propagation
R_2 = the ratio of second plastic zone propagation
$T(r)$ = temperature at radius r
T_y = yield temperature in reservoir subjected to temperature gradient
$T_{y,b}$ = yield temperature corresponding to onset of yielding at outer surface
$T^{NTG}(r_1)$ = specific negative temperature gradient
$T^{PTG}(r_1, r_2)$ = specific positive temperature gradient

Greek symbols

$\alpha(r)$ = coefficient of linear expansion at radius r
α_0 = material constant for coefficient of linear expansion
ΔT = difference between inner and outer temperatures
$\varepsilon_r(r)$ = radial strain
$\varepsilon_\theta(r)$ = circumferential strain
$\sigma_{eff}(r)$ = effective stress
$\sigma_r(r)$ = radial stress
$\sigma_r^e(r)$ = elastic radial stress
$\sigma_r^{p1}(r)$ = radial stress of first plastic zone
$\sigma_r^{p2}(r)$ = radial stress of second plastic zone
$\sigma_\theta(r)$ = circumferential stress
$\sigma_\theta^e(r)$ = elastic circumferential stress
$\sigma_\theta^{p1}(r)$ = circumferential stress of first plastic zone
$\sigma_\theta^{p2}(r)$ = circumferential stress of second plastic zone
$\sigma_y(r)$ = yield stress at radius r
σ_{y_0}= material constant for yield stress

چکیده

تحلیل ترموالاستوپلاستیک مخازن کروی هدفمند (FGM) به صورت تحلیلی انجام شده است. مخازن مزبور تحت گرادیان دمایی مثبت یا منفی و فشار داخلی به صورت مجزا و همزمان قرار گرفته‌اند. مدل توانی برای در نظر گرفتن تغییرات خواص مکانیکی در امتداد ضخامت به کار برده شده است. معیار تسلیم فون میزز و فرضیات مدل الاستیک–پلاستیک کامل برای بیان رفتار مصالح در ناحیه پلاستیک استفاده شده است.

الگوهای گسترش نواحی پلاستیک برای ترکیب‌های مختلف فشار داخلی و گرادیان دمایی مثبت یا منفی بررسی شده است. تحلیل عددی مسائل مشابه در نرم‌افزار آباکوس انطباق خوبی با نتایج تحلیلی نشان می‌دهد. کره‌ی هدفمند تحت فشار داخلی و گرادیان دمایی منفی در فشار کمتری نسبت به حالتی که تنها تحت فشار داخلی باشد، تسلیم می‌شود. ترکیب فشار داخلی و گرادیان دمایی منفی، بحرانی‌ترین ترکیب است. به علاوه نتایج نشان می‌دهند که اعمال یک گرادیان دمایی مثبت می‌تواند ظرفیت مخزن تحت فشار را افزایش دهد.