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Analysis of Squeezing Flow of Viscous Fluid under the Influence of Slip and Magnetic Field: Comparative Studies of Different Approximate Analytical Methods

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The various industrial and engineering applications of flow of fluid between parallel plates have continued to generate renewed interests. In this work, a comparative study of approximate analytical methods is carried out using differential transformation,homotopy perturbation, Adomian decomposition, variation of parameter and variational iteration methods for the analysis of a steady two-dimensional axisymmetric flow of an incompressible viscous fluid under the influence of a uniform transverse magnetic field with slip boundary condition. From the results, it is established that, the result of DTM and VPM shows to be more convenient for engineering calculations compared to the HPM as it appears more appealing than the HPM. Also, effects of pertinent flow, magnetic field and slip parameters are studied. By comparing the results of approximate analytical methods in this work with the numerical method using Runge-Kutta coupled with shooting method, the validity and the accuracy of approximate analytical solutions are established.

Keywords: First grade fluid; Squeezing flow; Slip boundary; Approximate analytical methods.

1 Introduction

The study of the flow of fluid between parallel plates has generated a lot of research interests due to its numerous applications in industrial and engineering applications such as moving pistons, chocolate fillers, hydraulic lifts, electric motors, flow inside syringes and nasogastric tubes, compression, and injection, power transmission squeezed film and polymers processing. In such applications, the flow of fluid are performed as a result of the moving apart or the coming together of two parallel plates. Following the pioneer work and the basic formulations of on squeezing flows under lubrication assumption by Stefan [1], there have been increasing research interests and many scientific studies on these types of flow. In a past work over few decades, Reynolds [2] analyzed the squeezing flow between elliptic plates while Archibald [3] investigated the same problem for rectangular plates.

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The earlier studies on squeezing flows were based on Reynolds equation which insufficiency for some cases has been shown by Jackson [4] and Usha and Sridharan [5].

Therefore, there have been several attempts and renewed research interests by different researchers to proper analyze and understand the squeezing flows [5-15]. In the past efforts to analyze such flow process, Rashidi et al. [16] used homotopy analysis method (HAM) to develop analytical approximate solutions to study the unsteady two dimensional axisymmetric squeezing flow between parallel plates while Duwairi et al. [17] investigated effects of squeezing on heat transfer of a viscous fluid between parallel plates. Qayyum et al. [18] studied the squeezing flow of non-Newtonian second grade fluids and micro-polar models presenting effect on velocity profiles. Hamdan [19] analyzed the effect of squeezing flow on dusty fluids discussing squeeze effect on fluid flow. Mahmood et al. [20] investigated the effects of Prandtl's number and Nusselt number on the squeezed flow and heat transfer over a porous surface for viscous fluids. Hatami and Jing [21] applied differential transformation method to study the natural convection of a non–Newtonian nanofluid between two vertical plates and Newtonian nanofluid between horizontal plates. Mohyud-Din et al. [22] investigated on heat and mass transfer analysis for the flow of a nanofluid between rotating parallel plates while Mohyud-Din and Khan [23] analyzed the nonlinear radiation effects on squeezing flow of a Casson fluid between parallel disks. Qayyum et al. [24] modeled and applied homotopy perturbation method to analyze the unsteady axisymmetric squeezing fluid flow through porous medium channel with slip boundary. Qayyum and Khan [25] presented the behavioral study of unsteady squeezing flow through porous medium using homotopy perturbation method. Mustafa et al. [26] presented their study on the heat and mass transfer in unsteady fluid flow under squeezed flow between two parallel plates using homotopy analysis method. In order to study the influence of magnefield on the sqeezing flow of non-Newtonian fluid, Siddiqui et al. [27] adopted homotopy perturbation method investigated the magnetic effect of squeezing viscous magnetohydrodynamics (MHD) fluid flow. Few years later, Domairry and Aziz [28] used homotopy perturbation method (HPM) to study the MHD squeezed flow between two parallel disks with suction or injection. Also, the effect of squeeze on Copper-water and Copper-Kerosene nanofluid between two parallel plates subjected to magnetic field was studied by Acharya et al [29] using the differential transformation method (DTM). Ahmed et al. [30] analyzed magneto hydrodynamic (MHD) squeezing flow of a Casson fluid between parallel disks. A year later, Ahmed et al. [31] investigated on MHD flow of an incompressible fluid through porous medium between dilating and squeezing permeable walls. The same year, Khan et al. [32] studied unsteady two-dimensional and axisymmetric squeezing flow between parallel plates. The same authors Khan et al. [33] MHD squeezing flow between two infinite plates while Hayat et al. [34] had earlier investigated the effect of squeezing flow of second grade fluid between two parallel disks. Khan et al. [35] analyzed unsteady squeezing flow of Casson fluid with magnetohydrodynamic effect and passing through porous medium while Ullah et al. [36] used homotopy perturbation method to present analytical solution of squeezing flow in porous medium with MHD effect. Thin Newtonian liquid films squeezing between two plates were studied by Grimm [37]. Squeezing flow under the influence of magnetic field is widely applied to bearing with liquid-metal lubrication [38–41].

Islam et al [42] studied squeezing fluid flow between the two infinite parallel plates in a porous medium channel. In case of many polymeric liquids when the weight of molecule is high, then they show slip at the boundary. The no-slip boundary condition is not applicable in this case. In many cases such as thin film problems, rarefied fluid problems, fluids containing concentrated suspensions, and flow on multiple interfaces, the no-slip boundary condition fails to work. Navier [43], for the first time, proposed the general boundary condition which demonstrates the fluid slip at the surface.

The difference of fluid velocity and velocity of the boundary is proportional to the shear stress at that boundary. The proportionality constant is named the slip parameter having length as its dimension. The slip condition is of great importance especially when fluids with elastic character are under consideration [44]. Newtonian fluid was considered by Ebaid [45] to study the effects of magnetic field and wall slip conditions on the peristaltic transport in an asymmetric channel. It has great importance in medical sciences, particularly in polishing artificial heart valves and internal cavities in many manufactured parts achieved by embedding such fluids with abrasives [46]. The influence of slip on the peristaltic motion of third-order fluid in asymmetric channel is studied by Hayat et al. [47]. The effects of slip condition on the rotating flow of a third grade fluid in a nonporous medium are investigated by Hayat and Abelman [48]. Their work was extended to a porous medium and obtaining the numerical solutions for the steady magnetohydrodynamics flow of a third grade fluid in a rotating frame is presented by Abelman et al. [49]. The past efforts in analyzing the squeezing flow problems have been largely based on the applications of approximate analytical methods. In this work, comparative analyses of DTM, HPM, ADM, VPM and VIM are carried out for MHD squeezing flow with slip boundary condition between two infinite plates approaching each other slowly. Also, the effects of the various flow parameters were investigated.

2 Problem Formulation

Consider, a squeezing flow of an incompressible Newtonian fluid with constant density ρ and viscosity μ, squeezed between two large planar parallel plates separated by a small distance $2h$ approaching each other with a low constant velocity v in the presence of a magnetic field, as shown in Figure (1).
Assume that the flow is quasi steady, the Navier-Stokes equations governing such flow when inertial terms are retained will be:

$$\nabla.\mathrm{v} = 0 \tag{1}$$

$$\rho\left[\frac{\partial \mathrm{v}}{\partial t} + (\nabla.\mathrm{v})\mathrm{v}\right] = \nabla.\mathrm{T} + (\mathrm{v}\times\mathrm{B})\times\mathrm{B} \tag{2}$$

Where ∇ denotes the material time derivatives, $\mathrm{T}$is the Cauchy stress tensor given by $\mathrm{T} = -pI + \mu A$ with $A = \nabla\mathrm{v} + (\mathrm{v})^t$.B is the total magnetic field given by $\mathrm{B} = \mathrm{B}_0 + b.\mathrm{B}_0$ and b represent the imposed and induced magnetic fields, respectively. The modified Ohm's law and Maxwell's equations. In the absence of displacement currents, are:

$$\mathrm{J} = \sigma\left[E + \mathrm{v}\times\mathrm{B}\right], \quad \nabla.\mathrm{B} = 0 \tag{3}$$

$$\nabla\times\mathrm{B} = \mu_m\mathrm{J},\ \mathrm{curl}E = \frac{\partial \mathrm{B}}{\partial t} \tag{4}$$

Here J is the electric current density, σ represents the electrical conductivity, E the electric field, and μ_m the magnetic permeability. If ρ, μ_m and σ are constant, b is negligible as compared to B_0, B is perpendicular to v so that the Reynold number is small with no electric field in the fluid flow region and then the magneto hydrodynamic force involved can be written as:

$$\mathrm{J}\times\mathrm{B} = -\sigma\mathrm{B}_0^2\mathrm{v} \tag{5}$$

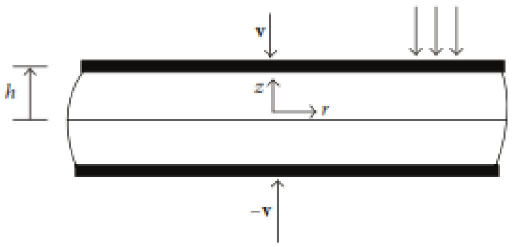

Figure 1 Model of the squeezing flow of viscous fluid under transverse uniform magnetic field

Assuming that the plates are non-conducting and the magnetic field is applied along the z-axis. The gap distance $2h$ between the plates changes slowly with time *t* for small values of the velocity **v** so that it can be taken as constant. An axisymmetric flow in cylindrical coordinates (r,θ,z) with *z*-axis perpendicular to plates and $z=\pm h$ at the plates. For axial symmetry, v is represented by $v=(v_r,0,v_z)$. In view of negligible body forces with no tangential velocity, Navier-Stokes equation [1, 6, 10] in cylindrical coordinates are:

$$\frac{\partial p}{\partial r}-\rho\Omega \mathrm{v}_r=-\mu\frac{\partial\Omega}{\partial z}-\sigma \mathrm{B}_0^2\mathrm{v}_r \tag{6}$$

$$\frac{\partial p}{\partial z}-\rho\Omega \mathrm{v}_r=\frac{\mu}{r}\frac{\partial}{\partial r}(r\Omega) \tag{7}$$

where

$$\Omega(r,z)=\frac{\partial \mathrm{v}_z}{\partial r}-\frac{\partial \mathrm{v}_r}{\partial z} \tag{8}$$

Introducing the stream function $\psi(r,z)$, we have

$$\mathrm{v}_r=\frac{1}{r}\frac{\partial\psi}{\partial z},\ \mathrm{v}_z=-\frac{1}{r}\frac{\partial\psi}{\partial r}. \tag{9}$$

Eliminating *p* from (9), we have

$$p\left[\frac{\partial\left(\psi,E^2\psi/r^2\right)}{\partial(r,z)}\right]=-\frac{\mu}{r}E^2\psi+\frac{\sigma \mathrm{B}_0^2}{r}\frac{\partial^2\psi}{\partial z^2} \tag{10}$$

where

$$E^2=\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2} \tag{11}$$

Using the transformation $\psi(r,z)=r^2 f(z)$, (10) can be written as

$$f^{(iv)}(z)-\frac{\sigma \mathrm{B}_0^2}{\mu}f''(z)+2\frac{\rho}{\mu}f(z)f'''(z)=0, \tag{12}$$

Subject to the slip boundary conditions

$$\begin{aligned}&f(0)=0,\ f''(0)=0,\\&f(h)=\frac{v}{2},\ f'(h)=\beta f''(h)\end{aligned} \tag{13}$$

The non-dimensional parameters are $F^* = g / v / 2,\ z^* = z / \mathrm{h},\ \mathrm{R}=\rho\mathrm{h}/\mu/\mathrm{v},$ and $\mathrm{m}=\mathrm{B}_0 h\sqrt{\sigma / \mu}$. Omitting the * for the sake of conveniences, (12) and (13) becomes

$$F^{(iv)}(z) - m^2 F''(z) + RF(z)F'''(z) = 0 \tag{15}$$

And the boundary conditions are

$$\begin{aligned} &F(0) = 0,\ \ \mathrm{F}''(0) = 0, \\ &F(1) = 1,\ \ \mathrm{F}'(1) = \gamma F''(1) \end{aligned} \tag{16}$$

With $\gamma = \beta / h$ and R, m are Reynolds and Hartmann numbers respectively [50].

3 Approximate Analytical Methods of Solution: Differential Transform Method

The differential transform method has widely been used to solve both singular and non-singular perturbed boundary values problems. It gives analytical solution to differential or integral solutions in the form of a polynomial by transforming each term in the differential equation or integral into a recursive form or relation of the equation which follows an iterative procedure for obtaining analytical series solutions of differential equation.

The basic definitions of the method is as follows

If $u(x)$ is analytic in the domain T, then it will be differentiated continuously with respect to space *x*.

$$\frac{d^p u(x)}{dx^p} = \varphi(x, p) \quad \text{for} \qquad \text{all} \ \ x \in T \tag{17}$$

For $x = x_i$, then $\varphi(x, p) = \varphi(x_i, p)$, where p belongs to the set of non-negative integers, denoted as the *p*-domain. Therefore Eq. (17) can be rewritten as

$$U(p) = \varphi(x_i, p) = \left[\frac{d^p u(x)}{dt^p}\right]_{x=x_i} \tag{18}$$

Where U_p is called the spectrum of $u(x)$ at $x = x_i$

If $u(x)$ can be expressed by Taylor's series, the $u(x)$ can be represented as

$$u(x) = \sum_{p}^{\infty} \left[\frac{(x - x_i)^p}{p!}\right] U(p) \tag{19}$$

Where Eq. (19) is called the inverse of $U(k)$ using the symbol 'D' denoting the differential transformation process and combining Eq. (18) and Eq. (19), it is obtained that

$$u(x) = \sum_{p=0}^{\infty} \left[\frac{(x - x_i)^p}{p!}\right] U(p) = D^{-1}U(p) \tag{20}$$

3.1 Operational properties of differential transformation method

If $u(x)$ *and* $v(x)$ are two independent functions with space (x) where $U(p)$ and $V(p)$ are the transformed function corresponding to $u(x)$ and $v(x)$, then it can be shown from the fundamental mathematics operations performed by differential transformation that.

i. If $z(x)=u(x)\pm v(x),$ then $Z(p)=U(p)\pm V(p)$

ii. If $z(x)=\alpha u(x),$ then $Z(p)=\alpha U(p)$

iii. If $z(x)=\dfrac{d^n u(x)}{dx^n},$ then $Z(p)=(p+1)\,(p+2)(p+3)...(p+n)U(p+n)$

iv. If $z(x)=u(x)v(x),$ then $Z(p)=\sum_{r=0}^{p}V(r)U(p-r)$

v. If $z(x)=u^m(x)$, then $Z(p)=\sum_{r=0}^{p}U^{m-1}(r)U(p-r)$

vi. If $z(x)=u(x)v(x),$ then $Z(p)=\sum_{r=0}^{p}(r+1)V(r+1)U(p-r)$

vii. If $z(x)=x^m\dfrac{d^n u(x)}{dx^n},$ then $Z(p)=\sum_{l=0}^{p}\delta(l-m-1)(p-l+1)(p-l+2)(p-l+3)...(p-l+n)U(p-l+n)$

viii. If $z(x)=\dfrac{du(x)}{dx}\dfrac{d^3u(x))}{dx^3},$ then $Z(p)=\sum_{l=0}^{p}U(p-l)(l+1)(l+2)(l+3)U(l+3)$

ix. If $z(x)=\dfrac{du(x)}{dx}\dfrac{d^2u(x))}{dx^2},$ then $Z(p)=\sum_{l=0}^{p}(p-l+1)U(p-l+1)(l+1)(l+2)U(l+2)$

x. If $z(x)=\left(\dfrac{du(x)}{dx}\right)^2$ then $Z(p)=\sum_{l=0}^{p}(p-l+1)U(p-l+1)(l+1)U(l+1)$

xi. If $z(x)=u\dfrac{du(x)}{dx},$ then $Z(p)=\sum_{l=0}^{p}U(p-l)(l+1)U(l+1)$

If $z(x)=\left[\dfrac{d^2u(x))}{dx^2}\right]^2$, then $Z(p)=\sum_{l=0}^{p}(p-l+1)(p-l+2)U(p-l+2)(l+1)(l+2)U(l+2)$

4 Application of the differential transform method to the present problem

The differential transform of (15) and (16) is given by

$$\begin{aligned}&(k+1)(k+2)(k+3)(k+4)F[k+4]-m^2\left((k+1)(k+2)F[k+2]\right)+\\&R\left(\sum_{l=0}^{k}(k-l+3)(k-l+2)(k-l+1)F[l]F[k-l+3]\right)=0\end{aligned}\tag{21}$$

With differential transformed boundary conditions

$$\tilde{F}[0]=0,\ \tilde{F}[1]=a,\ \tilde{F}[2]=0,\ \tilde{F}[3]=b,\tag{22}$$

$$\sum (k+1)F[k+1] = \gamma \sum (k+1)(k+2)F[k+2]$$

Where a and b are unknowns to be determined later using the boundary conditions of Eq. (16b).

Using Eqs. (21) and (22), the value of $\tilde{F}(i), i = 1,2,3,4,5,...19,20.$ are

$$\tilde{F}[4] = 0$$

$$\tilde{F}[5] = \frac{1}{20}\left(bm^2 - abR\right)$$

$$\tilde{F}[6] = 0$$

$$\tilde{F}[7] = \frac{1}{840}\left(bm^4 - 6b^2R - 4abm^2R + 3a^2bR^2\right)$$

$$\tilde{F}[8] = 0$$

$$\tilde{F}[9] = \left(\frac{bm^6 - 72b^2m^2R - 9abm^4R + 96ab^2R^2 + 23a^2bm^2R^2 - 15a^3bR^3}{60480}\right)$$

$$\tilde{F}[10] = 0$$

$$\tilde{F}[11] = \left(\frac{\begin{array}{l} bm^8 - 414b^2m^4R - 16abm^6R + 1296b^3R^2 + 1716ab^2m^2R^2 + \\ 86a^2bm^4R^2 - 1446a^2b^2R^3 - 176a^3bm^2R^3 + 105a^4bR^4 \end{array}}{6652800}\right)$$

$$\tilde{F}[12] = 0$$

$$\tilde{F}[13] = \left(\begin{array}{l} \frac{1}{1037836800}(bm^{10} - 1896b^2m^6R - 25abm^8R + \\ 43848b^3m^2R^2 + 14892ab^2m^4R^2 + 230a^2bm^6R^2 - 66456ab^3R^3 - \\ 35424a^2b^2m^2R^3 - 950a^3bm^4R^3 + 23580a^3b^2R^4 + 1689a^4bm^2R^4 - 945a^5bR^5) \end{array}\right)$$

$$\tilde{F}[14] = 0$$

$$\tilde{F}[15]=\left(\begin{array}{l}\frac{1}{217945728000}(bm^{12}-7974b^2m^8R-36abm^{10}R+703296b^3m^4R^2\\+98544ab^2m^6R^2+505a^2bm^8R^2-1362096b^4R^3-2874096ab^3m^2R^3-\\427716a^2b^2m^4R^3-3480a^3bm^6R^3+2540304a^2b^3R^4+753936a^3b^2m^2R^4+\\12139a^4bm^4R^4-428310a^4b^2R^5-19524a^5bm^2R^5+10395a^6bR^6)\end{array}\right)$$

$$\tilde{F}[16]=0$$

$$\tilde{F}[17]=\left(\begin{array}{l}\frac{1}{59281238016000}(bm^{14}-32472b^2m^{10}R-49abm^{12}R+8496576b^3m^6R^2\\+574104ab^2m^8R^2+973a^2bm^{10}R^2-93783744b^4m^2R^3-61494336ab^3m^4R^3-\\3844560a^2b^2m^6R^3-10045a^3bm^8R^3+151160256ab^4R^4+137564928a^2b^3m^2R^4\\+12050448a^3b^2m^4R^4+57379a^4bm^6R^4-90984960a^3b^3R^5-17320920a^4b^2m^2R^5\\-177331a^5bm^4R^5+8711640a^5b^2R^6+264207a^6bm^2R^6-135135a^7bR^7)\end{array}\right)$$

$$\tilde{F}[18]=0$$

$$\tilde{F}[19]=\left(\begin{array}{l}\frac{1}{20274183401472000}(bm^{16}-130686b^2m^{12}R-64abm^{14}R+89650368b^3m^8R^2\\+3121068ab^2m^{10}R^2+1708a^2bm^{12}R^2-3168258624b^4m^4R^3-978609024ab^3m^6R^3-\\29567250a^2b^2m^8R^3-24640a^3bm^{10}R^3+4090611456b^5R^4+12617990784ab^4m^2R^4+\\3791037312a^2b^3m^4R^4+141160488a^3b^2m^6R^4+208054a^4bm^8R^4-11321698752a^2b^4R^5\\-6076050048a^3b^3m^2R^5-354431586a^4b^2m^4R^5-1038016a^5bm^6R^5+3295339200a^4b^3R^6\\+436149036a^5b^2m^2R^6+2924172a^6bm^4R^6-198236430a^6b^2R^7-4098240a^7bm^2R^7+2027025a^8bR^8)\end{array}\right)$$

$$\tilde{F}[20]=0$$

According to the definition of DTM, the solution is

$$\begin{aligned}F(z)=&\tilde{F}[0]+z\tilde{F}[1]+z^2\tilde{F}[2]+z^3\tilde{F}[3]+z^4\tilde{F}[4]+z^5\tilde{F}[5]+z^6\tilde{F}[6]+z^7\tilde{F}[7]\\&+z^8\tilde{F}[8]+z^9\tilde{F}[9]+z^{10}\tilde{F}[10]+z^{11}\tilde{F}[11]++z^{12}\tilde{F}[12]+z^{13}\tilde{F}[13]\\&+z^{14}\tilde{F}[14]+z^{15}\tilde{F}[15]+z^{16}\tilde{F}[16]+z^{17}\tilde{F}[17]+z^{18}\tilde{F}[18]\\&+z^{19}\tilde{F}[19]+z^{20}\tilde{F}[20]\end{aligned}\tag{23}$$

5 Method of solution by homotopy perturbation method

It is very difficult to develop a closed-form solution for the above non-linear equation (19). Therefore, recourse has to be made to either approximation analytical method, semi-numerical method or numerical method of solution. In this work, homotopy perturbation method is used to solve the equation.

5.1 The basic idea of homotopy perturbation method

In order to establish the basic idea behind homotopy perturbation method, consider a system of nonlinear differential equations given as

$$A(U)-f(r)=0, \quad r\in\Omega \tag{24}$$

with the boundary conditions

$$B\left(u,\frac{\partial u}{\partial\eta}\right)=0, \qquad r\in\Gamma \tag{25}$$

where *A* is a general differential operator, *B* is a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω

The operator *A* can be divided into two parts, which are *L* and *N*, where *L* is a linear operator, *N* is a non-linear operator. Eq.(24) can be therefore rewritten as follows

$$L(u)+N(u)-f(r)=0 \tag{26}$$

By the homotopy technique, a homotopy $U(r,p):\Omega\times[0,1]\to R$ can be constructed, which satisfies

$$H(U,p)=(1-p)\left[L(U)-L(U_o)\right]+p\left[A(U)-f(r)\right]=0, \quad p\in[0,1] \tag{27}$$

Or

$$H(U,p)=L(U)-L(U_o)+pL(U_o)+p\left[N(U)-f(r)\right]=0 \tag{28}$$

In the above Eqs. (27) and (28), $p\in[0,1]$ is an embedding parameter, u_o is an initial approximation of equation of Eq. (24), which satisfies the boundary conditions.

Also, from Eqs. (27) and (28), we will have

$$H(U,0)=L(U)-L(U_o)=0 \tag{29}$$

$$H(U,0)=A(U)-f(r)=0 \tag{30}$$

The changing process of *p* from zero to unity is just that of $U(r,p)$ from $u_o(r)$ to $u(r)$. This is referred to homotopy in topology. Using the embedding parameter *p* as a small parameter, the solution of Eqs. (27) and (28) can be assumed to be written as a power series in p as given in Eq. (28)

$$U=U_o+pU_1+p^2U_2+\ldots \tag{31}$$

It should be pointed out that of all the values of *p* between 0 and 1, *p=1* produces the best result. Therefore, setting $p=1$, results in the approximation solution of Eq.(24)

$$u = \lim_{p \to 1} U = U_o + U_1 + U_2 + ... \tag{32}$$

The basic idea expressed above is a combination of homotopy and perturbation method. Hence, the method is called homotopy perturbation method (HPM), which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantages of the traditional perturbation techniques. The series Eq.(32) is convergent for most cases.

5.2 Application of the homotopy perturbation method to the present problem

According to homotopy perturbation method (HPM), one can construct an homotopy for Eq. (16) as

$$H(z,p) = (1-p)\tilde{F}^{(iv)} + p\left[\tilde{F}^{(iv)} - m^2\tilde{F}'' + R\tilde{F}\tilde{F}'''\right] \tag{33}$$

Using the embedding parameter p as a small parameter, the solution of Eqs. (16) can be assumed to be written as a power series in p as given in Eq. (33)

$$\tilde{F} = \tilde{F}_o + p\tilde{F}_1 + p^2\tilde{F}_2 + p^3\tilde{F}_3 + ... \tag{34}$$

On substituting Eqs. (34) and into Eq.(33) and expanding the equation and collecting all terms with the same order of p together, the resulting equation appears in form of polynomial in p . On equating each coefficient of the resulting polynomial in p to zero, we arrived at a set of differential equations and the corresponding boundary conditions as

$$p^0: \quad \tilde{F}_0^{(iv)} = 0, \quad \tilde{F}_0(0) = 0, \ \tilde{F}_0''(0) = 0, \quad \tilde{F}_0(1) = 1, \ \tilde{F}_0'(1) = \gamma\tilde{F}_0''(1) \tag{35}$$

$$p^1: \quad \tilde{F}_1^{(iv)} - m^2\tilde{F}_0'' + R\tilde{F}_0\tilde{F}_0''' = 0, \quad \tilde{F}_1(0) = 0, \ \tilde{F}_1''(0) = 0, \quad \tilde{F}_1(1) = 0, \ \tilde{F}_1'(1) = \gamma\tilde{F}_1''(1) \tag{36}$$

$$p^2: \quad \tilde{F}_2^{(iv)} - m^2\tilde{F}_1'' + R\tilde{F}_1\tilde{F}_0''' + R\tilde{F}_0\tilde{F}_1''' = 0, \ \tilde{F}_2(0) = 0, \ \tilde{F}_2''(0) = 0, \quad \tilde{F}_2(1) = 0, \ \tilde{F}_2'(1) = \gamma\tilde{F}_2''(1) \tag{37}$$

$$p^3: \tilde{F}_3^{(iv)} - m^2\tilde{F}_2'' + R\tilde{F}_2\tilde{F}_0''' + R\tilde{F}_1\tilde{F}_1''' + R\tilde{F}_0\tilde{F}_2''' = 0, \quad \tilde{F}_3(0) = 0, \ \tilde{F}_3''(0) = 0, \tilde{F}_3(1) = 0, \ \tilde{F}_3'(1) = \gamma\tilde{F}_3''(1) \tag{38}$$

$$\begin{gathered} p^4: \quad \tilde{F}_4^{(iv)} - m^2\tilde{F}_3'' + R\tilde{F}_3\tilde{F}_0''' + R\tilde{F}_3\tilde{F}_0''' + R\tilde{F}_2\tilde{F}_1''' + R\tilde{F}_1\tilde{F}_2''' + R\tilde{F}_0\tilde{F}_3''' = 0 \\ \tilde{F}_4(0) = 0, \ \tilde{F}_4''(0) = 0, \tilde{F}_4(1) = 0, \ \tilde{F}_4'(1) = \gamma\tilde{F}_4''(1) \end{gathered} \tag{39}$$

$$\begin{gathered} p^5: \quad \tilde{F}_5^{(iv)} - m^2\tilde{F}_4'' + R\tilde{F}_3\tilde{F}_0''' + R\tilde{F}_4\tilde{F}_0''' + R\tilde{F}_3\tilde{F}_1''' + R\tilde{F}_2\tilde{F}_2''' + R\tilde{F}_1\tilde{F}_3''' + R\tilde{F}_0\tilde{F}_4''' = 0 \\ \tilde{F}_5(0) = 0, \ \tilde{F}_5''(0) = 0, \tilde{F}_5(1) = 0, \ \tilde{F}_5'(1) = \gamma\tilde{F}_5''(1) \end{gathered} \tag{40}$$

On solving the above Eqs. (35-40), we arrived at

$$\tilde{F}_0(z) = \frac{3(2\gamma - 1)z + z^3}{2(3\gamma - 1)} \tag{41}$$

$$\tilde{F}_1(z)=\left\{\frac{3m^2}{3\gamma-1}+\frac{9R(2\gamma-1)}{2(3\gamma-1)^2}\right\}z^5+\frac{3R}{2(3\gamma-1)^2}z^7-\frac{1}{3(2\gamma+1)}\left\{\begin{array}{l}\gamma\left(\frac{60m^2}{3\gamma-1}+\frac{90R(2\gamma-1)}{(3\gamma-1)^2}+\frac{63R}{(3\gamma-1)^2}\right)\\-\left(\frac{12m^2}{3\gamma-1}+\frac{36R(2\gamma-1)}{(3\gamma-1)^2}+\frac{9R}{(3\gamma-1)^2}\right)\end{array}\right\}z^3$$

$$+\frac{1}{3(2\gamma+1)}\left\{\begin{array}{l}\gamma\left(\frac{60m^2}{3\gamma-1}+\frac{90R(2\gamma-1)}{(3\gamma-1)^2}+\frac{63R}{(3\gamma-1)^2}\right)\\-\left(\frac{12m^2}{3\gamma-1}+\frac{36R(2\gamma-1)}{(3\gamma-1)^2}+\frac{9R}{2(3\gamma-1)^2}\right)\end{array}\right\}z-\left\{\frac{3m^2}{3\gamma-1}+\frac{9R(2\gamma-1)}{(3\gamma-1)^2}+\frac{3R}{2(3\gamma-1)^2}\right\}z \tag{42}$$

In the same manner, the expressions for $\tilde{F}_2(z),\tilde{F}_3(z),\tilde{F}_4(z),\tilde{F}_5(z),\tilde{F}_6(z)...$ were obtained. However, they are too large expressions to be included in this paper.

Setting $p=1$, results in the approximation solution of Eq. (24)

$$F(z)=\lim_{p\to1}\tilde{F}(z)=\tilde{F}_o(z)+\tilde{F}_1(z)+\tilde{F}_2(z)+\tilde{F}_3(z)+\tilde{F}_4(z)+... \tag{43}$$

6.1 The basic idea of Adomian decomposition method (ADM)

Governing equations in terms of nonlinear differential equations are represented generally as

$$Lf+Nf+Rf+g=0. \tag{44}$$

Where *f* is the function to be decomposed, *Lf* is the highest linear part, *Nf* is the nonlinear part, *Rf* is the remainder and *g* is a known function.

The Adomian solution may be obtained from

$$\sum_{n=0}^{\infty}f_n=f_0-L^{-1}R\left(\sum_{n=0}^{\infty}f_n\right)-L^{-1}\left(\sum_{n=0}^{\infty}A_n\right)-L^{-1}g \tag{45}$$

Where A_n is the Adomian polynomial function as a result of the nonlinear term present in the model

$$A_n=\frac{1}{n!}\frac{d^n}{d\xi^n}\left(\left(\sum_{i=0}^{n}\xi^i f[i]\right)^{\zeta}\left(\sum_{i=0}^{n}\xi^i\frac{d^n}{d\eta^n}f[i]\right)\right)^{\varepsilon} \tag{46}$$

6.2 Application of ADM to the present problem

Given that the governing equation is

$$\frac{d^4}{dz^4}F(z)-m^2\frac{d^2}{dz^2}F(z)+RF(z)\frac{d^3}{dz^3}F(z)=0$$

With slip boundary conditions expressed as;

$$F(0)=0, \qquad F''(0)=0, \qquad F(1)=1, \qquad F(1)=yF''$$

For an order four differential equation, the leading term may be expressed with its corresponding constants as;

$$F_0 = a + bz + \frac{1}{2}cz^2 + \frac{1}{6}d\,z^3$$

and

$$\begin{aligned} F[i+1] &= L^{-1}\left(m^2 \frac{\mathrm{d}^2}{\mathrm{dz}^2} F(z)[i] - R\left(\sum_{n=0}^{\infty} A[i]\right)\right) \\ F[i+1] &= \int_0^z\int_0^z\int_0^z\int_0^z \left(m^2 \frac{\mathrm{d}^2}{\mathrm{dz}^2} F(z)[i] - R\left(\sum_{n=0}^{\infty} A[i]\right)\right) dz\,dz\,dz\,dz \end{aligned} \tag{47}$$

The constants will be obtained by solving the equations resulting from the use of the boundary conditions simultaneously. There is also need to generate an Adomian polynomial for the last term because it is nonlinear. For accuracy sake, 10 Adomian polynomial were generated as shown below:

$$\begin{aligned}
A_0 &= F_0(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_0(z) \\
A_1 &= F_0(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_1(z)+F_1(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_0(z) \\
A_2 &= F_0(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_2(z)+F_1(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_1(z)+F_2(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_0(z) \\
A_3 &= F_0(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_3(z)+F_1(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_2(z)+F_2(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_1(z)+F_3(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_0(z) \\
A_4 &= F_2(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_2(z)+F_1(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_3(z)+F_3(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_1(z)+F_4(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_0(z)+F_0(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_4(z) \\
A_5 &= F_0(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_5(z)+F_5(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_0(z)+F_3(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_2(z)+F_2(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_3(z)+F_1(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_4(z)+F_4(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_1(z) \\
A_6 &= F_0(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_6(z)+F_6(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_0(z)+F_1(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_5(z)+F_5(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_1(z)+F_3(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_3(z)+F_4(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_2(z)+F_2(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_4(z) \\
A_7 &= F_1(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_6(z)+F_6(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_1(z)+F_2(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_5(z)+F_5(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_2(z)+F_4(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_3(z)+F_3(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_4(z)+F_0(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_7(z)+F_7(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_0(z) \\
A_8 &= F_2(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_6(z)+F_6(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_2(z)+F_5(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_3(z)+F_3(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_5(z)+F_4(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_4(z)+F_1(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_7(z)+F_7(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_1(z)+F_0(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_8(z)+F_8(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_0(z) \\
A_9 &= F_0(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_9(z)+F_9(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_0(z)+F_3(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_6(z)+F_6(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_3(z)+F_5(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_4(z)+F_4(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_5(z)+F_2(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_7(z)+F_7(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_2(z)+F_1(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_8(z)+F_8(z)\frac{\mathrm{d}^3}{\mathrm{dz}^3}F_1(z)
\end{aligned} \tag{48}$$

Substituting Equation (48) into (47) and performing the simple iteration until a convergence is reached, the term by term solution is obtained as;

$$F_0 = a + bz + \frac{1}{2}cz^2 + \frac{1}{6}d\,z^3 \tag{49}$$

$$F_1 = -Rd\left(1/24az^4 + \frac{bz^5}{120} + \frac{cz^6}{720} + \frac{dz^7}{5040}\right) \tag{50}$$

$$F_2 = -R\left(\begin{array}{l} -\dfrac{Rd^3z^{11}}{1108800} - \dfrac{cRd^2z^{10}}{100800} + 1/9\left(-\dfrac{bRd^2}{2520} - \dfrac{c^2Rd}{4032}\right)z^9 \\ +1/8\left(-\dfrac{aRd^2}{840} - \dfrac{bRdc}{504}\right)z^8 + 1/7\left(-\dfrac{aRdc}{180} - \dfrac{b^2Rd}{240}\right)z^7 \\ -\dfrac{aRdbz^6}{240} - \dfrac{a^2Rdz^5}{120} \end{array}\right) \tag{51}$$

$$F_3 = -R\left\{\begin{array}{l} \dfrac{1051R^2d^4z^{15}}{217945728000} + \dfrac{1051cR^2d^3z^{14}}{14529715200} + 1/13\left(\begin{array}{l} \dfrac{bR^2d^3}{985600} + \dfrac{23c^2R^2d^2}{7983360} \\ -\dfrac{Rd}{7920}\left(-1/45bRd^2 - \dfrac{c^2Rd}{72}\right) \\ -\dfrac{Rd}{1320}\left(-\dfrac{bRd^2}{22680} - \dfrac{c^2Rd}{36288}\right) \end{array}\right)z^{13} \\ +1/12\left(\begin{array}{l} \dfrac{19aR^2d^3}{6652800} + \dfrac{31bR^2cd^2}{3326400} - \dfrac{cR}{1980}\left(-1/45bRd^2 - \dfrac{c^2Rd}{72}\right) \\ -\dfrac{Rd\left(-1/20aRd^2 - 1/12bRdc\right)}{5940} - \dfrac{Rd}{990}\left(-\dfrac{aRd^2}{6720} - \dfrac{bRdc}{4032}\right) \end{array}\right)z^{12} \\ +1/11\left(\begin{array}{l} \dfrac{13aR^2cd^2}{604800} - \dfrac{bR}{720}\left(-1/45bRd^2 - \dfrac{c^2Rd}{72}\right) \\ -\dfrac{cR\left(-1/20aRd^2 - 1/12bRdc\right)}{1440} - \dfrac{Rd\left(-1/6aRdc - 1/8b^2Rd\right)}{4320} \\ +\dfrac{R^2d^2b^2}{172800} - \dfrac{Rd}{720}\left(-\dfrac{aRdc}{1260} - \dfrac{b^2Rd}{1680}\right) \end{array}\right)z^{11} \\ +1/10\left(\begin{array}{l} -\dfrac{aR}{504}\left(-1/45bRd^2 - \dfrac{c^2Rd}{72}\right) \\ -\dfrac{bR\left(-1/20aRd^2 - 1/12bRdc\right)}{504} \\ -\dfrac{cR\left(-1/6aRdc - 1/8b^2Rd\right)}{1008} + \dfrac{R^2d^2ab}{4320} \end{array}\right)z^{10} + + \dfrac{a^2R^2dbz^7}{840} + \dfrac{a^3R^2dz^6}{720} \\ 1/9\left(\begin{array}{l} -\dfrac{aR\left(-1/20aRd^2 - 1/12bRdc\right)}{336} + \dfrac{R^2d^2a^2}{2520} \\ -\dfrac{bR\left(-1/6aRdc - 1/8b^2Rd\right)}{336} + \dfrac{aR^2cdb}{1344} \end{array}\right)z^9 + 1/8\left(\begin{array}{l} -\dfrac{aR\left(-1/6aRdc - 1/8b^2Rd\right)}{210} \\ +\dfrac{b^2R^2ad}{420} + \dfrac{cR^2a^2d}{840} \end{array}\right)z^8 \end{array}\right\} \tag{52}$$

The final ADM series solution after using the boundary conditions to obtained the constants becomes

$$F(z) = \sum_{i=0}^{\infty} F[i] \tag{53}$$

7.1 The basic idea of Variation of parameter method (VPM)

Nonlinear differential equations are represented generally as

$$Lf + Nf + Rf + g = 0\,. \tag{54}$$

Where f is the function to be decomposed, Lf is the highest linear part, Nf is the nonlinear part, Rf is the remainder and g is a known function.

The VPM iteration equation is given as;

$$f_{n+1}(\eta) = f_n(\eta) + \int_0^{\eta} \lambda \left[Rf_n(s) + Nf_n(s) - g(s) \right] \mathrm{d}s \tag{55}$$

with

$$\lambda = \frac{(\eta - s)^{n-1}}{(n-1)!} \tag{56}$$

Where λ is called the Wronskian parameter and n is the order of the differential equation under consideration.

7.2 Application of VPM to the present problem

Given that the governing equation is

$$\frac{\mathrm{d}^4}{\mathrm{d}z^4} F(z) - m^2 \frac{\mathrm{d}^2}{\mathrm{d}z^2} F(z) + RF(z) \frac{\mathrm{d}^3}{\mathrm{d}z^3} F(z) = 0 \tag{57}$$

With slip boundary conditions expressed as;

$$F(0) = 0, \qquad F''(0) = 0, \qquad F(1) = 1, \qquad F(1) = yF'' \tag{58}$$

Applying the principle of VPM,

$$F_{n+1} = F_n + \int_0^z \lambda \left(-m^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2} F(s) + RF(s) \frac{\mathrm{d}^3}{\mathrm{d}s^3} F(s) \right) \mathrm{d}s \tag{59}$$

Where the leading term is obtained from the single integration of the highest derivative as

$$F_0 = a + bz + \frac{1}{2} cz^2 + \frac{1}{6} d\, z^3 \tag{60}$$

Hence, the general series becomes;

$$F_{n+1} = a + bz + \frac{1}{2} cz^2 + \frac{1}{6} d\, z^3 + \int_0^z \lambda \left(-m^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2} F(s) + RF(s) \frac{\mathrm{d}^3}{\mathrm{d}s^3} F(s) \right) \mathrm{d}s \tag{61}$$

The term by term solution becomes;

$$F_0 = a + bz + \frac{1}{2}cz^2 + \frac{1}{6}d\,z^3 \tag{62}$$

$$F_1 = \left\{\begin{array}{l} a + bz + 1/2z^2c + 1/6z^3d + \dfrac{Rd^2z^7}{252} \\ +1/6\left(-1/12z\,Rd + 1/12Rc\right)dz^6 \\ +1/5\begin{pmatrix} 1/12z^2Rd \\ -1/4\eta zRc + 1/6Rb \end{pmatrix}dz^5 \\ +1/4\begin{pmatrix} -1/36z^3Rd + 1/4z^2Rc \\ -1/2z\,Rb + 1/6Ra \end{pmatrix}dz^4 \\ +1/3\begin{pmatrix} -1/12z^3Rc \\ +1/2z^2Rb \\ -1/2z\,Ra \end{pmatrix}dz^3 + \\ 1/2\begin{pmatrix} -1/6z^3Rb \\ +1/2z^2Ra \end{pmatrix}dz^2 - 1/6z^4Rad \end{array}\right\} \tag{63}$$

$$F(z) = a + bz + \frac{1}{2}cz^2 + \frac{1}{6}d\,z^3\left\{\begin{array}{l} a + bz + 1/2z^2c + 1/6z^3d + \dfrac{Rd^2z^7}{252} \\ +1/6\left(-1/12z\,Rd + 1/12Rc\right)dz^6 \\ +1/5\begin{pmatrix} 1/12z^2Rd \\ -1/4\eta zRc + 1/6Rb \end{pmatrix}dz^5 \\ +1/4\begin{pmatrix} -1/36z^3Rd + 1/4z^2Rc \\ -1/2z\,Rb + 1/6Ra \end{pmatrix}dz^4 \\ +1/3\begin{pmatrix} -1/12z^3Rc \\ +1/2z^2Rb \\ -1/2z\,Ra \end{pmatrix}dz^3 + \\ 1/2\begin{pmatrix} -1/6z^3Rb \\ +1/2z^2Ra \end{pmatrix}dz^2 - 1/6z^4Rad \end{array}\right\} + \ldots \tag{64}$$

The final VPM series solution after using the boundary conditions to obtained the constants becomes

$$F(z) = \sum_{i=0}^{\infty} F[i] \tag{65}$$

8.1 The basic idea of Variation of parameter method (VIM)

Nonlinear differential equations are represented generally as

$$Lf + Nf + Rf + g = 0. \quad (66)$$

Where f is the function to be decomposed, Lf is the highest linear part, Nf is the nonlinear part, Rf is the remainder and g is a known function.

The VIM iteration equation is given as;

$$f_{n+1}(\eta) = f_n(\eta) + \int_0^{\eta} \lambda \left[Lf_n(s) + Rf_n(s) + Nf_n(s) - g(s) \right] \mathrm{d}s \quad (67)$$

with

$$\lambda = \frac{(\eta - s)^{n-1}}{(n-1)!} \quad (68)$$

Where λ is called the Wronskian parameter and n is the order of the differential equation under consideration.

7.2 Application of VIM to the present problem

Given that the governing equation is

$$\frac{\mathrm{d}^4}{\mathrm{d}z^4} F(z) - m^2 \frac{\mathrm{d}^2}{\mathrm{d}z^2} F(z) + RF(z) \frac{\mathrm{d}^3}{\mathrm{d}z^3} F(z) = 0 \quad (69)$$

With slip boundary conditions expressed as;

$$F(0) = 0, \qquad F''(0) = 0, \qquad F(1) = 1, \qquad F(1) = yF'' \quad (70)$$

Applying the principle of VPM,

$$F_{n+1} = F_n + \int_0^z \lambda \left(\frac{\mathrm{d}^4}{\mathrm{d}s^4} F(s) - m^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2} F(s) + RF(s) \frac{\mathrm{d}^3}{\mathrm{d}s^3} F(s) \right) \mathrm{d}s \quad (71)$$

Where the leading term is obtained from the single integration of the highest derivative as

$$F_0 = a + bz + \frac{1}{2} cz^2 + \frac{1}{6} d z^3 \quad (72)$$

Hence, the general series becomes;

$$F_{n+1} = a + bz + \frac{1}{2} cz^2 + \frac{1}{6} d z^3 + \int_0^z \lambda \left(\frac{\mathrm{d}^4}{\mathrm{d}s^4} F(s) - m^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2} F(s) + RF(s) \frac{\mathrm{d}^3}{\mathrm{d}s^3} F(s) \right) \mathrm{d}s \quad (73)$$

The term by term solution becomes;

$$F_0 = a + bz + \frac{1}{2} cz^2 + \frac{1}{6} d z^3 \quad (74)$$

$$F_1 = \left\{ \begin{array}{l} a + bz + 1/2z^2c + 1/6z^3d \\ +1/6\left(-1/12zRd + 1/12Rc\right)dz^6 \\ +1/5\begin{pmatrix} 1/12z^2Rd \\ -1/4\eta zRc + 1/6Rb \end{pmatrix}dz^5 \\ +1/4\begin{pmatrix} -1/36z^3Rd + 1/4z^2Rc \\ -1/2zRb + 1/6Ra \end{pmatrix}dz^4 \\ +1/3\begin{pmatrix} -1/12z^3Rc \\ +1/2z^2Rb \\ -1/2zRa \end{pmatrix}dz^3 + +\dfrac{Rd^2z^7}{252} \\ 1/2\begin{pmatrix} -1/6z^3Rb \\ +1/2z^2Ra \end{pmatrix}dz^2 - 1/6z^4Rad \end{array} \right\} \tag{75}$$

$$F(z) = a + bz + \frac{1}{2}cz^2 + \frac{1}{6}d\,z^3 \left\{ \begin{array}{l} a + bz + 1/2z^2c + 1/6z^3d \\ +1/6\left(-1/12zRd + 1/12Rc\right)dz^6 \\ +1/5\begin{pmatrix} 1/12z^2Rd \\ -1/4\eta zRc + 1/6Rb \end{pmatrix}dz^5 \\ +1/4\begin{pmatrix} -1/36z^3Rd + 1/4z^2Rc \\ -1/2zRb + 1/6Ra \end{pmatrix}dz^4 \\ +1/3\begin{pmatrix} -1/12z^3Rc \\ +1/2z^2Rb \\ -1/2zRa \end{pmatrix}dz^3 + +\dfrac{Rd^2z^7}{252} \\ 1/2\begin{pmatrix} -1/6z^3Rb \\ +1/2z^2Ra \end{pmatrix}dz^2 - 1/6z^4Rad \end{array} \right\} + \ldots \tag{76}$$

The final VPM series solution after using the boundary conditions to obtained the constants becomes

$$F(z) = \sum_{i=0}^{\infty} F[i] \tag{77}$$

5 Results and Discussion

The above analysis shows a comparative study of approximate analytical methods of differential transformation and homotopy perturbation methods for the analysis of a steady two-dimensional axisymmetric flow of an incompressible viscous fluid under the influence of a uniform transverse magnetic field with slip boundary condition. Using DTM, a closed form series solution was obtained as it provides excellent approximations to the solution of the non-linear equation with higher accuracy than HPM. Also, the DTM shows to be more convenient for engineering calculations compared with the HPM as it appears more appealing than the HPM. However, due to the higher accuracy of DTM than HPM as shown the table (1), the solution of DTM is used to carry out the parametric study shown in Figs. (2).

Although, the DTM is somehow easier and straight-forward as compared to HPM, there is a rigour in developing the recursive relations or differential transforms coupled with the search for included unknown parameter that will satisfy second the boundary condition lead to additional computational cost in the generation of the solution to the problem using DTM. This drawback is not only peculiar to DTM, other approximate analytical methods such as HAM, ADM, VIM, DJM, TAM also required additional computational cost and time for the determination of included unknown parameter that will satisfy second the boundary condition. Also, the DTM has its own operational restrictions that severely narrow its functioning domain as it is limited to small domain. Using DTM for large or infinite domain is accompanied with either the application of before-treatment techniques such as domain transformation techniques, domain truncation techniques and conversion of the boundary value problems to initial value problems or the use of after-treatment techniques such as Pade-approximants, basis functions, cosine after-treatment technique, sine after-treatment technique and domain decomposition technique. This is because DTM was initially established for initial value problems. Amending the method to boundary value problems especially for large or infinite domains boundary value problems leads to unknown parameter that will satisfy second the boundary condition. This drawback in the other approximation analytical methods is not experienced in HPM as such tasks of before- and after-treatment techniques are not required in HPM as it easily applied to the boundary value problems without any included unknown parameter in the solution as found in DTM.

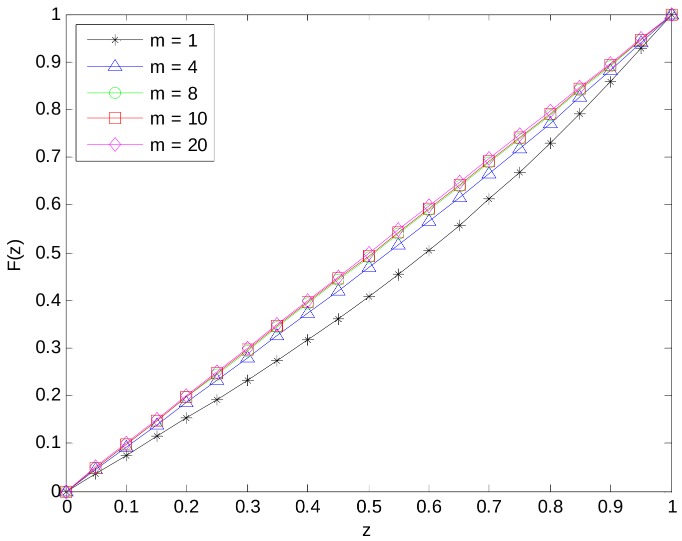

Figure 2 Effects of magnetic parameter on the flow behavior of the fluid under the influence of slip condition

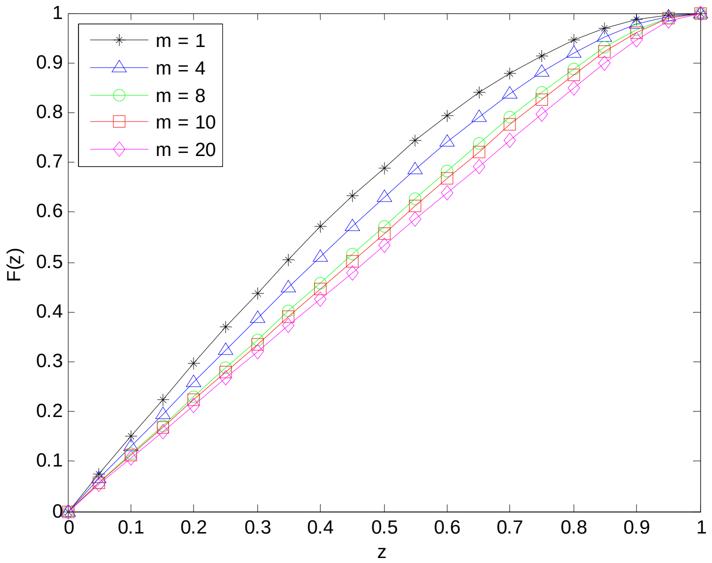

Figure 3 Effects of magnetic field parameter on the flow behavior of the fluid for no-slip condition

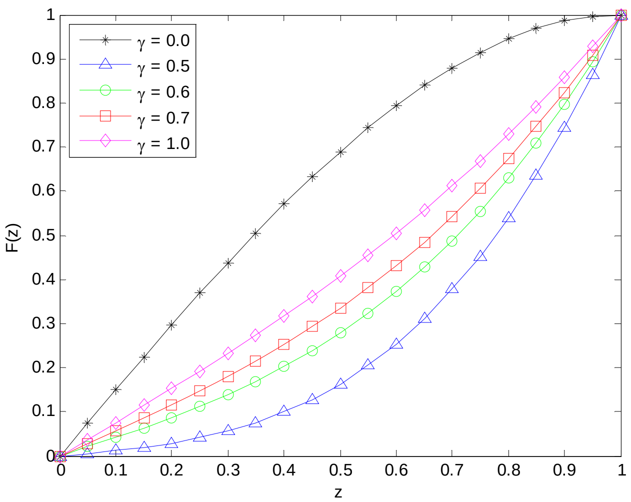

Figure 4 Effects of slip parameter on the flow behavior of the fluid

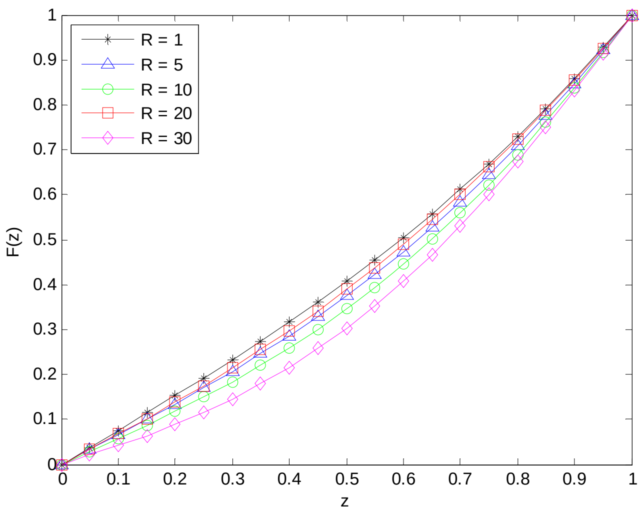

Figure 5 Effects of Reynolds number on the flow behavior of the fluid under the influence of slip condition

In order to get an insight into the problem, the effects of pertinent flow, magnetic field and slip parameters on the velocity profile of the fluid are investigated. Fig. (2) shows the effects of magnetic field parameter, Hartmann number m on the velocity of the fluid under the influence of slip condition, while Fig. (3) depicts the influence of the magnetic field parameter on the velocity of the fluid under no-slip condition. It could be inferred from the figures that the velocity of the fluid increases with increase in the magnetic parameter under slip condition while an opposite trend was recorded during no-slip condition as the velocity of the fluid decreases with increase in the magnetic field parameter under the no slip condition. Fig. (4) shows the influence of the slip parameter γ on the fluid velocity. By increasing γ, it is observed that the velocity of the fluid increases. Fig. (5) presents the effects of Reynold's number on the velocity of the fluid. It is observed from the figure that by increasing the value R, the velocity of the fluid decreases.

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